Solutions to Assignments 05

1. We first compute the variation of $\mathcal{E}[\phi] = \frac{1}{2}\int d^3x\ \left(\vec{\nabla}\phi\right)^2$:

$$\begin{aligned}\delta\mathcal{E}[\phi] &= \int d^3x\ \vec{\nabla}\phi\cdot\vec{\nabla}\delta\phi \\ &= \int d^3x\ \vec{\nabla}\cdot\left(\vec{\nabla}\phi\delta\phi\right) - \int d^3x\ (\Delta\phi)\,\delta\phi \\ &= -\int d^3x\ (\Delta\phi)\,\delta\phi\ , \qquad (1)\end{aligned}$$

where the first term in the second line vanishes because it is a boundary term and the variation is zero on the boundary. Finally, because for an extremum the variation has to vanish for any variation $\delta\phi$, we conclude

$$\delta\mathcal{E}[\phi] = 0\ \ \forall\ \delta\phi \quad \Leftrightarrow \quad \Delta\phi = 0\ . \qquad (2)$$

2. Again we compute the variation in exactly the same way

$$\begin{aligned}\delta S[\Phi_a] &= \int d^4x\ \left[-\eta^{\alpha\beta}\partial_\alpha\Phi^a\partial_\beta\delta\Phi^b\delta_{ab} - \frac{\partial V}{\partial\Phi^b}\delta\Phi^b\right] \\ &= \int d^4x\ \left[\delta_{ab}\Box\Phi^a - \frac{\partial V}{\partial\Phi^b}\right]\delta\Phi^b\ , \qquad (3)\end{aligned}$$

such that we can read off the equations of motion for the fields

$$\Box\Phi_b = \frac{\partial V}{\partial\Phi^b}\ , \qquad (4)$$

where $\Phi_b = \delta_{ab}\Phi^a \equiv \Phi^b$.

3. It is useful to first recall how this works in the case of classical mechanics (i.e. a 0+1 dimensional "field theory"). Consider a Lagrangian $L(q,\dot{q};t)$ that is a total time-derivative, i.e.

$$L(q,\dot{q};t) = \frac{d}{dt}F(q;t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q(t)}\dot{q}(t)\ \ . \qquad (5)$$

Then one has

$$\frac{\partial L}{\partial q(t)} = \frac{\partial^2 F}{\partial q(t)\partial t} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \qquad (6)$$

and

$$\frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial F}{\partial q(t)} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial^2 F}{\partial t\partial q(t)} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \qquad (7)$$

Therefore one has

$$\frac{\partial L}{\partial q(t)} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} \qquad \text{identically} \qquad (8)$$

Now we consider the field theory case. We define

$$\begin{aligned} L := \frac{d}{dx^\alpha} W^\alpha(\phi; x) &= \frac{\partial W^\alpha}{\partial x^\alpha} + \frac{\partial W^\alpha}{\partial \phi(x)} \frac{\partial \phi(x)}{\partial x^\alpha} \\ &= \partial_\alpha W^\alpha + \partial_\phi W^\alpha \partial_\alpha \phi \end{aligned} \tag{9}$$

to compute the Euler-Lagrange equations for it. One gets

$$\frac{\partial L}{\partial \phi} = \partial_\phi \partial_\alpha W^\alpha + \partial_\phi^2 W^\alpha \partial_\alpha \phi \tag{10}$$

$$\begin{aligned} \frac{d}{dx^\beta} \frac{\partial L}{\partial(\partial_\beta \phi)} &= \frac{d}{dx^\beta} \left(\partial_\phi W^\alpha \delta_\alpha^{\ \beta} \right) \\ &= \partial_\beta \partial_\phi W^\beta + \partial_\phi^2 W^\beta \partial_\beta \phi \ , \end{aligned} \tag{11}$$

and realizing that $\partial_\beta \partial_\phi W^\beta = \partial_\phi \partial_\beta W^\beta$ we have (10) = (11), thus the Euler-Lagrange equations are trivially satisfied.

Remark: One can also show the converse: if a Lagrangian L gives rise to Euler-Lagrange equations that are identically satisfied then (locally) the Lagrangian is a total derivative. The proof is simple. Assume that $L(q, \dot{q}; t)$ satisfies

$$\frac{\partial L}{\partial q} \equiv \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \tag{12}$$

identically. The left-hand side does evidently not depend on the acceleration $\ddot{q}$. The right-hand side, on the other hand, will in general depend on $\ddot{q}$ - unless L is at most linear in $\dot{q}$. Thus a necessary condition for L to give rise to identically satisfied Euler-Lagrange equations is that it is of the form

$$L(q, \dot{q}; t) = L^0(q; t) + L^1(q; t)\dot{q} \ . \tag{13}$$

Therefore

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{d}{dt} L^1 = \frac{\partial L^1}{\partial t} + \frac{\partial L^1}{\partial q} \dot{q} \tag{14}$$

and

$$\frac{\partial L}{\partial q} = \frac{\partial L^0}{\partial q} + \frac{\partial L^1}{\partial q} \dot{q} \ . \tag{15}$$

Noting that the 2nd terms of the previous two equations are equal, the Euler-Lagrange equations thus reduce to the condition

$$\frac{\partial L^1}{\partial t} = \frac{\partial L^0}{\partial q} \ . \tag{16}$$

This means that locally there is a function $F(q; t)$ such that

$$L^0 = \partial_t F \quad , \quad L^1 = \partial_q F \ , \tag{17}$$

and therefore

$$L = L^0 + L^1 \dot{q} = \partial_t F + \partial_q F \dot{q} = \frac{d}{dt} F \ , \tag{18}$$

as was to be shown. (Proof in the field theory case is analogous)

4. The Chern-Simons Action

First note that

$$
\begin{aligned}
S_{CS}[A] &= \int d^3x\, \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} \\
&= 2\int d^3x\, \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma\ .
\end{aligned} \tag{19}
$$

(a) To find the equations of motion one computes the variation

$$
\begin{aligned}
\delta S_{CS}[A] &= 2\int d^3x\, \epsilon^{\alpha\beta\gamma} \left(\delta A_\alpha \partial_\beta A_\gamma + A_\alpha \partial_\beta \delta A_\gamma\right) \\
&= 2\int d^3x\, \epsilon^{\alpha\beta\gamma} \left(\delta A_\alpha \partial_\beta A_\gamma - (\partial_\beta A_\alpha)\delta A_\gamma\right) \\
&= 2\int d^3x\, \epsilon^{\alpha\beta\gamma} \left(\partial_\beta A_\gamma - \partial_\gamma A_\beta\right)\delta A_\alpha \\
&= 2\int d^3x\, \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}\delta A_\alpha\ ,
\end{aligned} \tag{20}
$$

which implies

$$
\delta S_{CS}[A] = 0 \ \Leftrightarrow\ F_{\beta\gamma} = 0\ . \tag{21}
$$

(b) The equation of motion $F_{\alpha\beta} = 0$ is certainly gauge invariant. We now want to know if the Chern-Simons action $S_{CS}[A]$ is gauge invariant. To address the question we perform a gauge transformation $A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha\psi$ and see how the Chern-Simons term changes

$$
\begin{aligned}
S_{CS}[A'] &= S_{CS}[A] + \int d^3x\, \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}\partial_\alpha\psi \\
&= S_{CS}[A] + \int d^3x\, \partial_\alpha\left(\epsilon^{\alpha\beta\gamma} F_{\beta\gamma}\psi\right)
\end{aligned} \tag{22}
$$

using $\epsilon^{\alpha\beta\gamma}\partial_\alpha F_{\beta\gamma} = 0$. Thus, because the second term is a total derivative (i.e. the Lagrangian is invariant up to a total derivative), one sees that the Chern-Simons action is gauge invariant up to boundary terms.

(c) Combining the above result in (a) with the equations of motion of pure Maxwell theory, the result

$$
\partial_\alpha F^{\alpha\beta} + k\,\epsilon^{\beta\gamma\delta}F_{\gamma\delta} = 0\ \ . \tag{23}
$$

follows.

As an aside: the above theory with Lagrangian

$$L = L_m + k\ L_{cs} = -\tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \tfrac{1}{2}k\ \epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} \tag{24}$$

is also known as topologically massive Maxwell theory, since the CS term provides a "topological" gauge-invariant mass term for the photon (note that a mass term like $m^2 A_\alpha A^\alpha$, like in the KLein-Gordon equation, would not be gauge invariant - which is why it is usually claimed that the masslessness of the photon is due to gauge invariance).

One quick way to see this is to introduce the dual of the field strength

$$G^\beta = \tfrac{1}{2}\epsilon^{\beta\gamma\delta}F_{\gamma\delta} \tag{25}$$

in terms of which the equations of motion and the Bianchi identity take the form

$$\partial_\alpha G_\beta - \partial_\beta G_\alpha = 2k\ \epsilon_{\alpha\beta\gamma}G^\gamma \quad , \quad \partial_\beta G^\beta = 0 \tag{26}$$

respectively. Acting with ∂^α on the equation of motion and using the Bianchi identity and again the equation of motion one finds

$$\begin{aligned} \Box G_\beta &= 2k\ \epsilon_{\alpha\beta\gamma}\partial^\alpha G^\gamma = k\ \epsilon_{\alpha\beta\gamma}(\partial^\alpha G^\gamma - \partial^\gamma G^\alpha) \\ &= 2k^2\ \epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\gamma\delta}\ G_\delta = 4k^2 G_\beta \end{aligned} \tag{27}$$

so that the theory indeed describes excitations of mass $m^2 = 4k^2$.