Solutions to Assignments 05

1. We first compute the variation of $\mathcal{E}[\phi] = \frac{1}{2}\int d^3x\,\left(\vec{\nabla}\phi\right)^2$:

$$\begin{aligned}\delta\mathcal{E}[\phi] &= \int d^3x\,\vec{\nabla}\phi\cdot\vec{\nabla}\delta\phi \\ &= \int d^3x\,\vec{\nabla}\cdot\left(\vec{\nabla}\phi\delta\phi\right) - \int d^3x\,\left(\Delta\phi\right)\delta\phi \\ &= -\int d^3x\,\left(\Delta\phi\right)\delta\phi\,, \end{aligned} \tag{1}$$

where the first term in the second line vanishes because it is a boundary term and the variation is zero on the boundary. Finally, because for an extremum the variation has to vanish for any variation $\delta\phi$, we conclude

$$\delta\mathcal{E}[\phi] = 0\ \forall\,\delta\phi \quad\Leftrightarrow\quad \Delta\phi = 0\,. \tag{2}$$

2. Again we compute the variation in exactly the same way

$$\begin{aligned}\delta S[\Phi_a] &= \int d^4x\,\left[-\eta^{\alpha\beta}\partial_\alpha\Phi^a\partial_\beta\delta\Phi^b\delta_{ab} - \frac{\partial V}{\partial\Phi^b}\delta\Phi^b\right] \\ &= \int d^4x\,\left[\delta_{ab}\Box\Phi^a - \frac{\partial V}{\partial\Phi^b}\right]\delta\Phi^b\,, \end{aligned} \tag{3}$$

such that we can read off the equations of motion for the fields

$$\Box\Phi_b = \frac{\partial V}{\partial\Phi^b}\,, \tag{4}$$

where $\Phi_b = \delta_{ab}\Phi^a \equiv \Phi^b$.

3. It is useful to first recall how this works in the case of classical mechanics (i.e. a 0+1 dimensional "field theory"). Consider a Lagrangian $L(q,\dot{q};t)$ that is a total time-derivative, i.e.

$$L(q,\dot{q};t) = \frac{d}{dt}F(q;t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q(t)}\dot{q}(t)\quad. \tag{5}$$

Then one has

$$\frac{\partial L}{\partial q(t)} = \frac{\partial^2 F}{\partial q(t)\partial t} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \tag{6}$$

and

$$\frac{\partial L}{\partial\dot{q}(t)} = \frac{\partial F}{\partial q(t)} \quad\Rightarrow\quad \frac{d}{dt}\frac{\partial L}{\partial\dot{q}(t)} = \frac{\partial^2 F}{\partial t\partial q(t)} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \tag{7}$$

Therefore one has

$$\frac{\partial L}{\partial q(t)} = \frac{d}{dt}\frac{\partial L}{\partial\dot{q}(t)} \qquad \text{identically} \tag{8}$$

Now we consider the field theory case. We define

$$
\begin{aligned}
L := \frac{d}{dx^\alpha} W^\alpha(\phi; x) &= \frac{\partial W^\alpha}{\partial x^\alpha} + \frac{\partial W^\alpha}{\partial \phi(x)} \frac{\partial \phi(x)}{\partial x^\alpha} \\
&= \partial_\alpha W^\alpha + \partial_\phi W^\alpha \partial_\alpha \phi
\end{aligned} \tag{9}
$$

to compute the Euler-Lagrange equations for it. One gets

$$
\frac{\partial L}{\partial \phi} = \partial_\phi \partial_\alpha W^\alpha + \partial_\phi^2 W^\alpha \partial_\alpha \phi \tag{10}
$$

$$
\begin{aligned}
\frac{d}{dx^\beta} \frac{\partial L}{\partial(\partial_\beta \phi)} &= \frac{d}{dx^\beta} \left(\partial_\phi W^\alpha \delta_\alpha^{\ \beta} \right) \\
&= \partial_\beta \partial_\phi W^\beta + \partial_\phi^2 W^\beta \partial_\beta \phi \ ,
\end{aligned} \tag{11}
$$

and realizing that $\partial_\beta \partial_\phi W^\beta = \partial_\phi \partial_\beta W^\beta$ we have (10) = (11), thus the Euler-Lagrange equations are trivially satisfied.

Remark: One can also show the converse: if a Lagrangian L gives rise to Euler-Lagrange equations that are identically satisfied then (locally) the Lagrangian is a total derivative. The proof is simple. Assume that $L(q, \dot{q}; t)$ satisfies

$$
\frac{\partial L}{\partial q} \equiv \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \tag{12}
$$

identically. The left-hand side does evidently not depend on the acceleration $\ddot{q}$. The right-hand side, on the other hand, will in general depend on $\ddot{q}$ - unless L is at most linear in $\dot{q}$. Thus a necessary condition for L to give rise to identically satisfied Euler-Lagrange equations is that it is of the form

$$
L(q, \dot{q}; t) = L^0(q; t) + L^1(q; t)\dot{q} \ . \tag{13}
$$

Therefore

$$
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{d}{dt} L^1 = \frac{\partial L^1}{\partial t} + \frac{\partial L^1}{\partial q} \dot{q} \tag{14}
$$

and

$$
\frac{\partial L}{\partial q} = \frac{\partial L^0}{\partial q} + \frac{\partial L^1}{\partial q} \dot{q} \ . \tag{15}
$$

Noting that the 2nd terms of the previous two equations are equal, the Euler-Lagrange equations thus reduce to the condition

$$
\frac{\partial L^1}{\partial t} = \frac{\partial L^0}{\partial q} \ . \tag{16}
$$

This means that locally there is a function $F(q; t)$ such that

$$
L^0 = \partial_t F \quad , \quad L^1 = \partial_q F \ , \tag{17}
$$

and therefore

$$
L = L^0 + L^1 \dot{q} = \partial_t F + \partial_q F \dot{q} = \frac{d}{dt} F \ , \tag{18}
$$

as was to be shown. (Proof in the field theory case is analogous)