KFT Solutions 03

1. Inhomogeneous Maxwell-Equations and Potentials

(a) Under the gauge transformation $A_\beta \to A_\beta + \partial_\beta \Psi$, $F_{\alpha\beta}$ transforms as

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha \to \partial_\alpha A_\beta + \partial_\alpha \partial_\beta \Psi - \partial_\beta A_\alpha - \partial_\beta \partial_\alpha \Psi = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{1}$$

and therefore $F_{\alpha\beta}$ is gauge-invariant.

(b) With $A_\alpha = (-\phi/c, \vec{A})$ one has

$$\begin{aligned} F_{0k} &= -F_{k0} = \partial_0 A_k - \partial_k A_0 = c^{-1}(\partial_t A_k + \partial_k \phi) = -E_k/c \\ F_{ik} &= \partial_i A_k - \partial_k A_i = \epsilon_{ik\ell} B_\ell \quad (F_{12} = B_3 \quad \text{etc.}) \end{aligned} \tag{2}$$

and therefore, with $F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}$,

$$F^{0k} = -F^{k0} = -F_{0k} = E_k/c \quad , \quad F^{ik} = F_{ik} = \epsilon_{ik\ell} B_\ell \ . \tag{3}$$

(c) Thus, with $J^\alpha = (\rho c, \vec{J})$ one has

$$\partial_\alpha F^{\alpha 0} = \partial_k F^{k0} = -c^{-1} \vec{\nabla}.\vec{E} = -\rho/(\epsilon_0 c) = -\mu_0 c \rho = -\mu_0 J^0 \tag{4}$$

and

$$\begin{aligned} \partial_\alpha F^{\alpha 1} &= \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = c^{-2} \partial_t E_1 - \partial_2 B_3 + \partial_3 B_2 \\ &= -(\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E})_1 = -\mu_0 J_1 = -\mu_0 J^1 \end{aligned} \tag{5}$$

(and likewise for the 2- and 3-components).

(d) One has

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = \Box A^\beta - \partial^\beta \partial_\alpha A^\alpha = -\mu_0 J^\beta \tag{6}$$

and therefore $\Box A_\beta - \partial_\beta \partial_\alpha A^\alpha = -\mu_0 J_\beta$.

(e) One has

$$\Box(A_\beta + \partial_\beta \Psi) - \partial_\beta \partial_\alpha (A^\alpha + \partial^\alpha \Psi) = \Box A_\beta + \partial_\beta \Box \Psi - \partial_\beta \partial_\alpha A^\alpha - \partial_\beta \Box \Psi \tag{7}$$

Since the $\Box\Psi$-term cancels, the expression is gauge invariant.

(f) From $\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$ one deduces $-\mu_0 \partial_\beta J^\beta = \partial_\beta \partial_\alpha F^{\alpha\beta} = 0$ because $F^{\alpha\beta} = -F^{\beta\alpha}$ is anti-symmetric while $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ is symmetric.