KFT Solutions 04

1. The Homogeneneous Maxwell-Equations

   (a) One has $\partial_\alpha F_{\beta\gamma} = \partial_\alpha \partial_\beta A_\gamma - \partial_\alpha \partial_\gamma A_\beta$ etc. Using the fact that 2nd partial derivatives commute one deduces

   $$\begin{aligned} & \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} \\ = \; & \partial_\alpha \partial_\beta A_\gamma - \partial_\gamma \partial_\alpha A_\beta + \partial_\gamma \partial_\alpha A_\beta - \partial_\beta \partial_\gamma A_\alpha + \partial_\beta \partial_\gamma A_\alpha - \partial_\alpha \partial_\beta A_\gamma = 0 \end{aligned} \tag{1}$$

   (b) $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

      i. Two indices equal ($\alpha = \beta$, say): $\partial_\alpha F_{\alpha\gamma} + \partial_\alpha F_{\gamma\alpha} + \partial_\gamma F_{\alpha\alpha} = 0$ is identically satisfied because $F_{\alpha\alpha} = 0$, $F_{\alpha\gamma} + F_{\gamma\alpha} = 0$.

      ii. All 3 indices spatial, $(\alpha, \beta, \gamma) = (1, 2, 3)$:

      $$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{21} = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \vec{\nabla}.\vec{B} = 0 \tag{2}$$

      iii. One index time, the others spatial, e.g. $(\alpha, \beta, \gamma) = (0, 1, 2)$:

      $$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = c^{-1}(\partial_t B_3 + \partial_1 E_2 - \partial_2 E_1) = c^{-1}(\vec{\nabla} \times \vec{E} + \partial_t \vec{B})_3 = 0 \tag{3}$$

      (and likewise for the other components).

2. The dual field strength tensor

   The dual field strength tensor is defined by

   $$\tilde{F}^{\alpha\beta} = \tfrac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad . \tag{4}$$

   With $F_{0i} = -c^{-1} E_i$, $F_{ij} = \epsilon_{ijk} B_k$ and $\tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$ one gets for the non-zero components of $\tilde{F}^{\alpha\beta}$:

   $$\tilde{F}^{0i} = \tfrac{1}{2} \epsilon^{0i\gamma\delta} F_{\gamma\delta} = \tfrac{1}{2} \epsilon^{0ijk} F_{jk} = \tfrac{1}{2} \epsilon^{0ijk} \epsilon_{jkl} B_l = -B_i \tag{5}$$

   $$\tilde{F}^{ij} = \tfrac{1}{2} \epsilon^{ij\gamma\delta} F_{\gamma\delta} = \epsilon^{ij0k} F_{0k} = -c^{-1} \epsilon^{ij0k} E_k = c^{-1} \epsilon^{ijk} E_k \; . \tag{6}$$

   Note that $\tilde{F}^{\alpha\beta}$ can be obtained from $F^{\alpha\beta}$ by the replacement $\vec{B} \to \vec{E}/c$, $\vec{E}/c \to -\vec{B}$ ("electro-magnetic duality transformation").

   The equation $\partial_\lambda \tilde{F}^{\lambda\delta} = 0$ can then be written as

   $$\partial_\lambda \tilde{F}^{\lambda 0} = -\partial_i \tilde{F}^{0i} = \vec{\nabla} \cdot \vec{B} = 0 \tag{7}$$

   $$\begin{aligned} \partial_\lambda \tilde{F}^{\lambda j} &= \partial_0 \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = -c^{-1} \partial_t B_j - c^{-1} \epsilon_{jik} \partial_i E_k \\ &= -\frac{1}{c} \left(\partial_t \vec{B} + \vec{\nabla} \times \vec{E} \right)_j = 0 \quad , \end{aligned} \tag{8}$$

   which proves the assertion.

3. Lorentz Invariants

(a)

$$\begin{aligned} I_1 &= \tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = \tfrac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) \\ &= \tfrac{1}{2}\left(-(F_{0i})^2 + (F_{ij})^2\right) = \tfrac{1}{2}\left(\vec{B}^2 - c^{-2}\vec{E}^2\right) \qquad (9) \\ I_2 &= \tfrac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} = \tfrac{1}{4}\left(F_{0i}\tilde{F}^{0i} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij}\right) \\ &= \tfrac{1}{4}\left(2c^{-1}E_i B_i + \epsilon_{ijk} B_k c^{-1} \epsilon^{ijl} E_l\right) = c^{-1}\vec{E}\cdot\vec{B} \qquad (10) \end{aligned}$$

where one has used $\epsilon^{ijl}\epsilon_{ijk} = 2\delta^l_k$. If $\vec{E} = 0$ in one inertial system, then $I_1 > 0$ and $I_2 = 0$ in all inertial systems, and thus $\vec{E}.\vec{B} = 0$ and $|\vec{E}| < |\vec{B}|$ in all inertial systems.

(b)

$$8I_2 = \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} = 2\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha A_\beta) F_{\gamma\delta} = 2\partial_\alpha\left(\epsilon^{\alpha\beta\gamma\delta} A_\beta F_{\gamma\delta}\right) \qquad (11)$$

because $\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\gamma\delta} = 0$ (Bianchi identity). Thus I_2 is a total derivative, $I_2 = \partial_\alpha C^\alpha$. C^α is not gauge invariant but changes by a total derivative under a gauge transformation, because

$$\epsilon^{\alpha\beta\gamma\delta} A_\beta F_{\gamma\delta} \to \epsilon^{\alpha\beta\gamma\delta} A_\beta F_{\gamma\delta} + \epsilon^{\alpha\beta\gamma\delta}\partial_\beta\psi F_{\gamma\delta} \qquad (12)$$

and

$$\epsilon^{\alpha\beta\gamma\delta}\partial_\beta\psi F_{\gamma\delta} = \partial_\beta(\epsilon^{\alpha\beta\gamma\delta}\psi F_{\gamma\delta}) \qquad (13)$$

(again by the Bianchi identity).