Solutions to Assignments 06

1. Noether Current and Noether Energy-Momentum Tensor for Field Theories

   (a) A rotation in the Φ_1, Φ_2 field space is

   $$\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \to \begin{pmatrix}\cos\gamma & \sin\gamma\\ -\sin\gamma & \cos\gamma\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \tag{1}$$

   Therefore an infinitesimal rotation in the Φ_1, Φ_2 field space reads

   $$\Delta\Phi_1 = \gamma\Phi_2 \quad , \quad \Delta\Phi_2 = -\gamma\Phi_1 \; . \tag{2}$$

   Under such a transformation, the potential $V(\Phi_1^2 + \Phi_2^2)$ is trivially invariant and the kinetic term can also be seen to be invariant,

   $$\Delta S_{\text{kin}} = -\frac{1}{2}\int d^4x \; 2\left(\partial_\alpha\Phi_1\partial^\alpha\gamma\Phi_2 - \partial_\alpha\gamma\Phi_1\partial^\alpha\Phi_2\right) = 0 \quad \Rightarrow \quad \Delta S = 0 \; . \tag{3}$$

   The current in given by

   $$J^\alpha_\Delta = \frac{\partial L}{\partial(\partial_\alpha\Phi_a)}\Delta\Phi_a = -\gamma\left(\Phi_1\partial^\alpha\Phi_2 - \Phi_2\partial^\alpha\Phi_1\right) \tag{4}$$

   and for a solution Φ_a to the equations of motions

   $$\Box\Phi_a = \frac{\partial V}{\partial\Phi_a} = 2V'(\Phi_1^2 + \Phi_2^2)\Phi_a \tag{5}$$

   one can explicitly check that

   $$\begin{aligned}\partial_\alpha J^\alpha_\Delta &= -\gamma\left(\partial_\alpha\Phi_1\partial^\alpha\Phi_2 + \Phi_1\Box\Phi_2 - \partial_\alpha\Phi_2\partial^\alpha\Phi_1 - \Phi_2\Box\Phi_1\right)\\ &= -2\gamma\left(\Phi_1\Phi_2 - \Phi_2\Phi_1\right)V'(\Phi_1^2 + \Phi_2^2) = 0 \; .\end{aligned} \tag{6}$$

   (b) The canonical (Noether) Energy-Momentum Tensor (or Stress-Energy tensor) is given by

   $$\theta^\alpha{}_\beta = -\partial^\alpha\Phi_a\partial_\beta\Phi_a + \delta^\alpha{}_\beta\left(\frac{1}{2}\partial^\gamma\Phi_a\partial_\gamma\Phi_a + V(\Phi_a)\right) \; . \tag{7}$$

   It is conserved for Φ_a a solution to the equations of motion $\Box\Phi_a = \frac{\partial V}{\partial\Phi_a}$:

   $$\begin{aligned}\partial_\alpha\theta^\alpha{}_\beta &= -\Box\Phi_a\partial_\beta\Phi_a - \partial^\alpha\Phi_a\partial_\alpha\partial_\beta\Phi_a + \partial^\gamma\Phi_a\partial_\beta\partial_\gamma\Phi_a + \partial_\beta V(\Phi_a)\\ &= -\frac{\partial V}{\partial\Phi_a}\partial_\beta\Phi_a + \partial_\beta V(\Phi_a) = 0 \; .\end{aligned} \tag{8}$$

2. The Maxwell Energy-Momentum Tensor

(a) We rewrite the covariant energy-momentum tensor as

$$\begin{aligned} T^{\alpha}_{\ \beta} &= -F^{\alpha\gamma}F_{\beta\gamma} + \tfrac{1}{4}\delta^{\alpha}_{\ \beta}\,F^2 = -F^{\alpha\gamma}\partial_\beta A_\gamma + \tfrac{1}{4}\delta^{\alpha}_{\ \beta}\,F^2 + F^{\alpha\gamma}\partial_\gamma A_\beta \\ &= \Theta^{\alpha}_{\ \beta} + \partial_\gamma(F^{\alpha\gamma}A_\beta) - (\partial_\gamma F^{\alpha\gamma})A_\beta \ \ . \end{aligned} \tag{9}$$

The 2nd term is identically conserved, $\partial_\alpha\partial_\gamma(F^{\alpha\gamma}A_\beta) \equiv 0$ because of symmetry / anti-symmetry, and the 3rd term is zero for a solution to the sourceless Maxwell equations.

(b) Since the Lagrangian L is gauge invariant, one has

$$\tilde{\Delta}L = \Delta L = \partial_\mu(\epsilon^\mu L) \tag{10}$$

and therefore

$$\frac{\partial L}{\partial(\partial_\alpha A_\gamma)}\tilde{\Delta}A_\gamma - \epsilon^\alpha L = (-F^{\alpha\gamma}F_{\mu\gamma} + \tfrac{1}{4}\delta^{\alpha}_{\mu}F^2)\epsilon^\mu = T^{\alpha}_{\ \mu}\epsilon^\mu \ \ . \tag{11}$$

(c) The trace of $T^{\alpha}_{\ \beta}$ is

$$T^{\alpha}_{\ \alpha} = -F^{\alpha\gamma}F_{\alpha\gamma} + \tfrac{1}{4}\delta^{\alpha}_{\ \alpha}\,F_{\gamma\delta}F^{\gamma\delta} = -F^{\alpha\gamma}F_{\alpha\gamma} + F_{\gamma\delta}F^{\gamma\delta} = 0\ . \tag{12}$$

Then one shows that $T_{\alpha\beta}$ is symmetric computing

$$T_{\alpha\beta} = \eta_{\alpha\gamma}T^{\gamma}_{\ \beta} = -F_{\alpha}^{\ \rho}F_{\beta\rho} + \tfrac{1}{4}\eta_{\alpha\beta}F_{\rho\delta}F^{\rho\delta}\ , \tag{13}$$

and using the fact that $\eta_{\alpha\beta}$ and $F_{\alpha}^{\ \rho}F_{\beta\rho} = F_{\alpha\rho}F_{\beta}^{\ \rho} = F_{\beta}^{\ \rho}F_{\alpha\rho}$ are symmetric.

(d) The component $T^{0}_{\ 0}$ is

$$\begin{aligned} T^{0}_{\ 0} &= -F^{0i}F_{0i} + \tfrac{1}{4}F_{\gamma\delta}F^{\gamma\delta} \\ &= -(c^{-1}E^i)(-c^{-1}E_i) + \tfrac{1}{2}(B^2 - c^{-2}E^2) = \frac{1}{2}\left(E^2/c^2 + B^2\right)\ , \end{aligned} \tag{14}$$

where $\frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}$ has been computed in assignments 04 (exercise 3). The component $T^{k}_{\ 0}$ is

$$T^{k}_{\ 0} = -F^{k\gamma}F_{0\gamma} = -F^{kj}F_{0j} = \epsilon_{kji}B_i c^{-1}E_j = c^{-1}\epsilon_{kji}E_jB_i = S_k/c = S^k/c \tag{15}$$

(e) One easily computes (Lines 1+2)

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= -F^{\beta}_{\ \gamma}\partial_\alpha F^{\alpha\gamma} - F^{\alpha\gamma}\partial_\alpha F^{\beta}_{\ \gamma} + \tfrac{1}{2}\eta^{\alpha\beta}F^{\gamma\delta}\partial_\alpha F_{\gamma\delta} \\ &= \eta^{\beta\lambda}F^{\gamma\delta}\partial_\delta F_{\lambda\gamma} + \tfrac{1}{2}\eta^{\lambda\beta}F^{\gamma\delta}\partial_\lambda F_{\gamma\delta} \end{aligned} \tag{16}$$

where the Maxwell equation $\partial_\alpha F^{\alpha\gamma} = 0$ was used in the 1st term, and some indices have been relabelled in the 2nd term to make it more similar to the 3rd term. Now we can use the antisymmetry of $F^{\gamma\delta}$ to rewrite the 2nd term as (Line 3)

$$\eta^{\beta\lambda}F^{\gamma\delta}\partial_\delta F_{\lambda\gamma} = \tfrac{1}{2}\eta^{\beta\lambda}F^{\gamma\delta}(\partial_\delta F_{\lambda\gamma} - \partial_\gamma F_{\lambda\delta})\ \ . \tag{17}$$

Plugging this into the previous result and using the homogeneous Maxwell equations, one finds (Line 4)

$$\partial_\alpha T^{\alpha\beta} = \tfrac{1}{2}\eta^{\lambda\beta}F^{\gamma\delta}\left(\partial_\lambda F_{\gamma\delta} + \partial_\delta F_{\lambda\gamma} + \partial_\gamma F_{\delta\lambda}\right) = 0\ . \tag{18}$$