1. It is useful to first recall how this works in the case of classical mechanics (i.e. a 0+1 dimensional “field theory”). Consider a Lagrangian $L(q, \dot{q}; t)$ that is a total time-derivative, i.e.

$$L(q, \dot{q}; t) = \frac{d}{dt} F(q; t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \dot{q}(t)} \dot{q}(t) .$$

Then one has

$$\frac{\partial L}{\partial q(t)} = \frac{\partial^2 F}{\partial q(t) \partial t} + \frac{\partial^2 F}{\partial q(t)^2} \dot{q}(t) \quad (2)$$

and

$$\frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial F}{\partial q(t)} \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial^2 F}{\partial t \partial q(t)} + \frac{\partial^2 F}{\partial q(t)^2} \dot{q}(t) \quad (3)$$

Therefore one has

$$\frac{\partial L}{\partial q(t)} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} \quad \text{identically} \quad (4)$$

Now we consider the field theory case. We define

$$L := \frac{d}{dx^\alpha} W^\alpha(\phi; x) = \frac{\partial W^\alpha}{\partial x^\alpha} + \frac{\partial W^\alpha}{\partial \phi(x)} \frac{\partial \phi}{\partial x^\alpha} = \partial_\alpha W^\alpha + \partial_\phi W^\alpha \partial_\alpha \phi \quad (5)$$

to compute the Euler-Lagrange equations for it. One gets

$$\frac{\partial L}{\partial \phi} = \partial_\phi \partial_\alpha W^\alpha + \partial_\phi^2 W^\alpha \partial_\alpha \phi \quad (6)$$

$$\frac{d}{dx^\beta} \frac{\partial L}{\partial (\partial_\beta \phi)} = \frac{d}{dx^\beta} \left( \partial_\phi W^\alpha \delta^\beta_\alpha \right) = \partial_\beta \partial_\phi W^\beta + \partial_\phi^2 W^\beta \partial_\beta \phi . \quad (7)$$

Thus (since partial derivatives commute) the Euler-Lagrange equations are satisfied identically.

**Remark:** One can also show the converse: if a Lagrangian $L$ gives rise to Euler-Lagrange equations that are identically satisfied then (locally) the Lagrangian is a total derivative. The proof is simple. Assume that $L(q, \dot{q}; t)$ satisfies

$$\frac{\partial L}{\partial q} \equiv \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \quad (8)$$

identically. The left-hand side does evidently not depend on the acceleration $\ddot{q}$. The right-hand side, on the other hand, will in general depend on $\ddot{q}$ - unless $L$ is at most linear in $\dot{q}$. Thus a necessary condition for $L$ to give rise to identically satisfied Euler-Lagrange equations is that it is of the form

$$L(q, \ddot{q}; t) = L^0(q; t) + L^1(q; t)\ddot{q} . \quad (9)$$
Therefore
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{d}{dt} L^1 = \frac{\partial L^1}{\partial t} + \frac{\partial L^1}{\partial q} \dot{q} \tag{10} \]
and
\[ \frac{\partial L}{\partial q} = \frac{\partial L^0}{\partial q} + \frac{\partial L^1}{\partial q} \dot{q} \tag{11} \]
Noting that the 2nd terms of the previous two equations are equal, the Euler-Lagrange equations thus reduce to the condition
\[ \frac{\partial L^1}{\partial t} = \frac{\partial L^0}{\partial q} \tag{12} \]
This means that locally there is a function \( F(q; t) \) such that
\[ L^0 = \partial_t F, \quad L^1 = \partial_q F \tag{13} \]
and therefore
\[ L = L^0 + L^1 \dot{q} = \partial_t F + \partial_q F \dot{q} = \frac{d}{dt} F, \tag{14} \]
as was to be shown. (Proof in the field theory case is analogous)

2. Complex Scalar Field I: Action and Equations of Motion

The action is
\[ S[\Phi] = \int d^4x \left( -\frac{1}{2} \xi^{\alpha \beta} \partial_\alpha \Phi \partial_\beta \Phi^* - W(\Phi, \Phi^*) \right) \]
\[ = \int d^4x \left( -\frac{1}{2} \xi^{\alpha \beta} \partial_\alpha \phi_1 \partial_\beta \phi_1 - \frac{1}{2} \xi^{\alpha \beta} \partial_\alpha \phi_2 \partial_\beta \phi_2 - V(\phi_1, \phi_2) \right) \tag{15} \]
(a) Varying \( \phi_1 \) in the 2nd line, while keeping \( \phi_2 \) fixed, one finds
\[ \delta S = \int d^4x \left( -\xi^{\alpha \beta} \partial_\alpha \phi_1 \partial_\beta \delta \phi_1 - (\partial W/\partial \phi_1) \delta \phi_1 \right). \tag{16} \]
Integrating by parts the 1st term, one obtains
\[ (1/2) \Box \phi_1 = \partial V/\partial \phi_1. \]
Analogously for \( \phi_2 \).

(b) Using
\[ \partial V/\partial \phi_1 = (\partial W/\partial \Phi)(\partial \Phi/\partial \phi_1) + (\partial W/\partial \Phi^*)(\partial \Phi^*/\partial \phi_1) = (\partial W/\partial \Phi) + (\partial W/\partial \Phi^*) \]
\[ \partial V/\partial \phi_2 = (\partial W/\partial \Phi^*)(\partial \Phi^*/\partial \phi_2) + (\partial W/\partial \Phi)(\partial \Phi/\partial \phi_2) = i(\partial W/\partial \Phi) - i(\partial W/\partial \Phi^*) \tag{17} \]
one finds
\[ \Box \Phi = \Box \phi_1 + i \Box \phi_2 = \partial V/\partial \phi_1 + i \partial V/\partial \phi_2 = 2\partial W/\partial \Phi^* \]
\[ \Box \Phi^* = \Box \phi_1 - i \Box \phi_2 = \partial V/\partial \phi_1 - i \partial V/\partial \phi_2 = 2\partial W/\partial \Phi \tag{18} \]
(c) Varying only \( \Phi^* \) in the first line of the action, while keeping \( \Phi \) fixed, one finds
\[ \delta S = \int d^4x \left( -\frac{1}{2} \xi^{\alpha \beta} \partial_\alpha \Phi \partial_\beta \delta \Phi^* - (\partial W/\partial \Phi^*) \delta \Phi^* \right). \tag{19} \]
Integrating by parts the 1st term, one obtains \( (1/2) \Box \Phi \), and thus the correct Euler-Lagrange equation for \( \Phi \) (analogously for \( \Phi \leftrightarrow \Phi^* \)).
3. Complex Scalar Field II: Phase Invariance and Noether-Theorem

(a) If the potential is a function of $\Phi^*\Phi$, both the potential and the derivative terms of the Lagrangian are obviously invariant under

$$\Phi(x) \rightarrow e^{i\theta(x)} \Phi(x) \quad , \quad \Phi^*(x) \rightarrow e^{-i\theta(x)} \Phi^*(x)$$

for constant $\theta$, since in this case the derivatives transform the same way, i.e.

$$\partial_\alpha \Phi(x) \rightarrow e^{i\theta} \partial_\alpha \Phi(x) \quad , \quad \partial_\alpha \Phi^*(x) \rightarrow e^{-i\theta} \partial_\alpha \Phi^*(x)$$

(b) Infinitesimally, one has

$$\Delta \Phi = i\theta \Phi \quad , \quad \Delta \Phi^* = -i\theta \Phi^*$$

and therefore the corresponding Noether current is

$$J^{\alpha}_\Delta = \frac{\partial L}{\partial (\partial_\alpha \Phi)} \Delta \Phi + \frac{\partial L}{\partial (\partial_\alpha \Phi^*)} \Delta \Phi^* = -(i\theta/2)(\Phi \partial^\alpha \Phi^* - \Phi^* \partial^\alpha \Phi)$$

where (as usual) $\partial^\alpha = \eta^{\alpha\beta} \partial_\beta$. Calculating its divergence, one finds (ignoring the irrelevant constant prefactor, and using the equations of motion)

$$\partial_\alpha (\Phi \partial^\alpha \Phi^* - \Phi^* \partial^\alpha \Phi) = \partial_\alpha \Phi \partial^\alpha \Phi^* + \Phi \Box \Phi^* - \partial_\alpha \Phi^* \partial^\alpha \Phi - \Phi^* \Box \Phi = \Phi \Box \Phi^* - \Phi^* \Box \Phi = 2(\Phi \partial W / \partial \Phi - \Phi^* \partial W / \partial \Phi^*)$$

This is not (and should not be) zero in general, but it is zero precisely when $W = W(\Phi^*\Phi)$. Indeed, in that case one has

$$\partial W(\Phi^*\Phi) / \partial \Phi = W'(\Phi^*\Phi) \Phi^* \quad , \quad \partial W(\Phi^*\Phi) / \partial \Phi^* = W'(\Phi^*\Phi) \Phi$$

and therefore

$$\Phi \partial W / \partial \Phi - \Phi^* \partial W / \partial \Phi^* = W'(\Phi^*\Phi) (\Phi \Phi^* - \Phi^* \Phi) = 0$$

4. Complex Scalar Field III: Gauge Invariance and Minimal Coupling

(a) Under

$$\Phi(x) \rightarrow e^{i\theta(x)} \Phi(x) \quad , \quad \Phi^*(x) \rightarrow e^{-i\theta(x)} \Phi^*(x) \quad , \quad A_\alpha(x) \rightarrow A_\alpha(x) + \partial_\alpha \theta(x)$$

the partial derivative transforms as

$$\partial_\alpha \Phi \rightarrow \partial_\alpha (e^{i\theta} \Phi) = e^{i\theta} (\partial_\alpha \Phi + i(\partial_\alpha \theta) \Phi)$$

Therefore the covariant derivative

$$D_\alpha \Phi = \partial_\alpha \Phi - iA_\alpha \Phi \quad , \quad D_\alpha \Phi^* = \partial_\alpha \Phi^* + iA_\alpha \Phi^*$$
transforms as
\[ D_\alpha \Phi \rightarrow e^{i\theta} (\partial_\alpha \Phi + i(\partial_\alpha \theta)\Phi) - ie^{i\theta} A_\alpha \Phi - ie^{i\theta} (\partial_\alpha \theta)\Phi \]
\[ = e^{i\theta} (\partial_\alpha \Phi - iA_\alpha \Phi) = e^{i\theta} D_\alpha \Phi \]  (30)

Likewise
\[ D_\alpha \Phi^* \rightarrow e^{-i\theta} D_\alpha \Phi^*. \]  (31)

(b) It is now obvious that the action
\[ S[\Phi, A] = \int d^4x \left( -\frac{1}{2} \eta^{\alpha\beta} D_\alpha \Phi D_\beta \Phi^* - W(\Phi\Phi^*) \right) \]  (32)

is gauge invariant.

(c) The action is
\[ S = S_{\text{Maxwell}}[A] + S[\Phi, A] = \int d^4x (-\frac{1}{4} F^2) + S[\Phi, A]. \]  (33)

The equations of motion for \( \Phi \) and \( \Phi^* \) are simply the covariant versions of
the equations of motion from Exercise 2, namely
\[ D^\alpha D_\alpha \Phi = 2 \partial W/\partial \Phi^*, \quad D^\alpha D_\alpha \Phi^* = 2 \partial W/\partial \Phi. \]  (34)

Variation with respect to \( A \) leads to
\[ \delta S = \int d^4x \left( \partial_\alpha F^{\alpha\beta} + J^\beta \right) \delta A_\beta \]  (35)

where
\[ J^\beta = (i/2) \left( \Phi D^\beta \Phi^* - \Phi^* D^\beta \Phi \right) \]  (36)

The equations of motion \( \partial_\alpha F^{\alpha\beta} + J^\beta = 0 \) imply (and therefore require) that \( \partial_\beta J^\beta = 0 \). Let us show that this equation is satisfied as a consequence of the equations of motion for \( \Phi \).

First of all, we have
\[ \partial_\beta (\Phi D^\beta \Phi^*) = \partial_\beta \Phi D^\beta \Phi^* + \Phi \partial_\beta D^\beta \Phi^*. \]  (37)

Adding and subtracting \( +iA_\beta \Phi \), we can write this as
\[ \partial_\beta (\Phi D^\beta \Phi^*) = D_\beta \Phi D^\beta \Phi^* + \Phi D_\beta D^\beta \Phi^*. \]  (38)

Since the first term is invariant under the exchange \( \Phi \leftrightarrow \Phi^* \), one finds
\[ \partial_\beta \left( \Phi D^\beta \Phi^* - \Phi^* D^\beta \Phi \right) = \Phi D_\beta D^\beta \Phi^* - \Phi^* D_\beta D^\beta \Phi \]  (39)

Note that this is just the covariant version of the divergence of the Noether current in Exercise 3, and the remaining step in the proof that this vanishes for a solution to the equations of motion is now identical to that in Exercise 3.