Solutions to Assignments 06

1. The Maxwell Energy-Momentum Tensor

   (a) We rewrite the covariant energy-momentum tensor as

   $$\begin{aligned} T^{\alpha}_{\ \beta} &= -F^{\alpha\gamma}F_{\beta\gamma} + \tfrac{1}{4}\delta^{\alpha}_{\ \beta}\,F^2 = -F^{\alpha\gamma}\partial_\beta A_\gamma + \tfrac{1}{4}\delta^{\alpha}_{\ \beta}\,F^2 + F^{\alpha\gamma}\partial_\gamma A_\beta \\ &= \Theta^{\alpha}_{\ \beta} + \partial_\gamma(F^{\alpha\gamma}A_\beta) - (\partial_\gamma F^{\alpha\gamma})A_\beta \ \ . \end{aligned} \qquad (1)$$

   The 2nd term is identically conserved, $\partial_\alpha\partial_\gamma(F^{\alpha\gamma}A_\beta) \equiv 0$ because of symmetry / anti-symmetry, and the 3rd term is zero for a solution to the sourceless Maxwell equations.

   (b) $\Delta A_\alpha = \epsilon^\mu \partial_\mu A_\alpha$ evidently implies that

   $$\Delta F(A) = \epsilon^\mu \partial_\mu F(A) \qquad (2)$$

   for any tensorial object $F(A)$ constructed from A. For $F_{\alpha\beta}$ one can also explicitly (but unnecessarily) verify this from

   $$\Delta F_{\alpha\beta} = \partial_\alpha \Delta A_\beta - \partial_\beta \Delta A_\alpha = \epsilon^\mu(\partial_\alpha\partial_\mu A_\beta - \partial_\beta\partial_\mu A_\alpha) = \epsilon^\mu\partial_\mu F_{\alpha\beta} \ \ . \qquad (3)$$

   In particular, for the Lagrangian $L(A) = -F^{\alpha\beta}F^{\alpha\beta}/4$, which does not depend explicitly on the coordinates x, one has

   $$\Delta L(A) = \frac{d}{dx^\mu}(\epsilon^\mu L) \ \ . \qquad (4)$$

   Since $\tilde{\Delta}A$ differs from $\Delta(A)$ by a gauge transformation, its action on any gauge-invariant tensorial object constructed from A agrees with that of ΔA,

   $$F(A) \quad \text{gauge invariant} \quad \Rightarrow \quad \tilde{\Delta}F(A) = \Delta F(A) = \epsilon^\mu\partial_\mu F(A) \qquad (5)$$

   For $F_{\alpha\beta}$ one can also explicitly (but unnecessarily) verify this from

   $$\tilde{\Delta}F_{\alpha\beta} = \partial_\alpha\tilde{\Delta}A_\beta - \partial_\beta\tilde{\Delta}A_\alpha = \epsilon^\mu(\partial_\alpha F_{\mu\beta} - \partial_\beta F_{\mu\alpha}) = \epsilon^\mu\partial_\mu F_{\alpha\beta} \ \ , \qquad (6)$$

   where we made use of the Bianchi identity. In particular, since $L(A)$ is gauge invariant one has

   $$\tilde{\Delta}L(A) = \Delta L(A) = \frac{d}{dx^\mu}(\epsilon^\mu L) \ \ , \qquad (7)$$

   and one can apply Noether's theorem to $\tilde{\Delta}$.

   (c) Doing this one obtains the quadruplet of conserved currents

   $$\tilde{J}^\alpha = \frac{\partial L}{\partial(\partial_\alpha A_\gamma)}\tilde{\Delta}A_\gamma - \epsilon^\alpha L = (-F^{\alpha\gamma}F_{\mu\gamma} + \tfrac{1}{4}\delta^\alpha_\mu F^2)\epsilon^\mu = T^{\alpha}_{\ \mu}\epsilon^\mu \ \ , \qquad (8)$$

   and thus one finds directly the correct covariant energy-momentum tensor of Maxwell theory.

(d) The trace of $T^\alpha_{\ \beta}$ is

$$T^\alpha_{\ \alpha} = -F^{\alpha\gamma}F_{\alpha\gamma} + \tfrac{1}{4}\delta^\alpha_{\ \alpha}\, F_{\gamma\delta}F^{\gamma\delta} = -F^{\alpha\gamma}F_{\alpha\gamma} + F_{\gamma\delta}F^{\gamma\delta} = 0\ . \tag{9}$$

Then one shows that $T_{\alpha\beta}$ is symmetric computing

$$T_{\alpha\beta} = \eta_{\alpha\gamma}T^\gamma_{\ \beta} = -F_\alpha^{\ \rho}F_{\beta\rho} + \tfrac{1}{4}\eta_{\alpha\beta}F_{\rho\delta}F^{\rho\delta}\ , \tag{10}$$

and using the fact that $\eta_{\alpha\beta}$ and $F_\alpha^{\ \rho}F_{\beta\rho} = F_{\alpha\rho}F_\beta^{\ \rho} = F_\beta^{\ \rho}F_{\alpha\rho}$ are symmetric.

(e) The component $T^0_{\ 0}$ is

$$\begin{aligned} T^0_{\ 0} &= -F^{0i}F_{0i} + \tfrac{1}{4}F_{\gamma\delta}F^{\gamma\delta} \\ &= -(c^{-1}E^i)(-c^{-1}E_i) + \tfrac{1}{2}(B^2 - c^{-2}E^2) = \frac{1}{2}\left(E^2/c^2 + B^2\right)\ , \end{aligned} \tag{11}$$

where $\frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}$ has been computed in assignments 04 (exercise 3). The component $T^k_{\ 0}$ is

$$T^k_{\ 0} = -F^{k\gamma}F_{0\gamma} = -F^{kj}F_{0j} = \epsilon_{kji}B_i c^{-1}E_j = c^{-1}\epsilon_{kji}E_jB_i = S_k/c = S^k/c \tag{12}$$

(f) One easily computes (Lines 1+2)

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= -F^\beta_{\ \gamma}\partial_\alpha F^{\alpha\gamma} - F^{\alpha\gamma}\partial_\alpha F^\beta_{\ \gamma} + \tfrac{1}{2}\eta^{\alpha\beta}F^{\gamma\delta}\partial_\alpha F_{\gamma\delta} \\ &= \eta^{\beta\lambda}F^{\gamma\delta}\partial_\delta F_{\lambda\gamma} + \tfrac{1}{2}\eta^{\lambda\beta}F^{\gamma\delta}\partial_\lambda F_{\gamma\delta} \end{aligned} \tag{13}$$

where the Maxwell equation $\partial_\alpha F^{\alpha\gamma} = 0$ was used in the 1st term, and some indices have been relabelled in the 2nd term to make it more similar to the 3rd term. Now we can use the antisymmetry of $F^{\gamma\delta}$ to rewrite the 2nd term as (Line 3)

$$\eta^{\beta\lambda}F^{\gamma\delta}\partial_\delta F_{\lambda\gamma} = \tfrac{1}{2}\eta^{\beta\lambda}F^{\gamma\delta}(\partial_\delta F_{\lambda\gamma} - \partial_\gamma F_{\lambda\delta})\ . \tag{14}$$

Plugging this into the previous result and using the homogeneous Maxwell equations, one finds (Line 4)

$$\partial_\alpha T^{\alpha\beta} = \tfrac{1}{2}\eta^{\lambda\beta}F^{\gamma\delta}\left(\partial_\lambda F_{\gamma\delta} + \partial_\delta F_{\lambda\gamma} + \partial_\gamma F_{\delta\lambda}\right) = 0\ . \tag{15}$$