Solutions 06

1. Noether Energy-Momentum Tensor

   The claim follows from

$$\begin{aligned}
\frac{d}{dx^a}\Theta^a_{\ b} &= \frac{d}{dx^a}\left(-\frac{\partial L}{\partial(\partial_a\Phi^A)}\partial_b\Phi^A + \delta^a_b L\right) \\
&= -\left(\frac{d}{dx^a}\frac{\partial L}{\partial(\partial_a\Phi^A)}\right)\partial_b\Phi^A - \frac{\partial L}{\partial(\partial_a\Phi^A)}\partial_a\partial_b\Phi^A + \frac{d}{dx^b}L \\
&= \left(\frac{\partial L}{\partial\Phi^A} - \frac{d}{dx^a}\frac{\partial L}{\partial(\partial_a\Phi^A)}\right)\partial_b\Phi^A + \frac{\partial}{\partial x^b}L
\end{aligned} \tag{1}$$

2. Noether Energy-Momentum Tensor for a Scalar Field

   The action is

$$\begin{aligned}
S[\phi] &= \int d^4x\left(-\tfrac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - V(\phi)\right) \equiv \int d^4x\left(-\tfrac{1}{2}(\partial\phi)^2 - V(\phi)\right) \\
\Rightarrow \quad \delta S[\phi] &= \int d^4x\left(-\eta^{ab}\partial_a\phi\partial_b\delta\phi - V'(\phi)\delta\phi\right) \\
&= \int d^4x\left(\eta^{ab}\partial_a\partial_b\phi - V'(\phi)\right)\delta\phi = \int d^4x\left(\Box\phi - V'(\phi)\right)\delta\phi \\
\Rightarrow \quad \Box\phi &= V'(\phi)
\end{aligned} \tag{2}$$

   The energy-momentum tensor is

$$\Theta_{ab} = -\frac{\partial L}{\partial(\partial^a\phi)}\partial_b\phi + \eta_{ab}L = \partial_a\phi\partial_b\phi - \eta_{ab}(\tfrac{1}{2}(\partial\phi)^2 + V(\phi)) \quad . \tag{3}$$

   The claim follows from

$$\begin{aligned}
\partial^a\Theta_{ab} &= (\Box\phi)\partial_b\phi + \partial_a\phi\partial^a\partial_b\phi - \partial_b(\tfrac{1}{2}(\partial\phi)^2 + V(\phi)) \\
&= (\Box\phi)\partial_b\phi + \partial_a\phi\partial^a\partial_b\phi - \partial_b\partial^a\phi\partial_a\phi - V'(\phi)\partial_b\phi \\
&= (\Box\phi - V'(\phi))\partial_b\phi \quad .
\end{aligned} \tag{4}$$

3. Maxwell Energy-Momentum Tensor

$$T_{ab} = F_{ac}F_b^{\ c} - \tfrac{1}{4}\eta_{ab}F_{cd}F^{cd} \quad . \tag{5}$$

   (a) see the Lecture Notes

   (b)

$$\Delta_T A_c = \delta_T A_c + \partial_c(\epsilon^b A_b) = -\epsilon^b F_{bc} \quad , \tag{6}$$

   is a variation, and therefore

$$\Delta_T F_{cd} = \partial_c\Delta_T A_d - \partial_d\Delta_T A_c = -\epsilon^b(\partial_c F_{bd} - \partial_d F_{bc}) \quad . \tag{7}$$

The Bianchi identity implies

$$\partial_c F_{bd} - \partial_d F_{bc} = -\partial_c F_{db} - \partial_d F_{bc} = +\partial_b F_{cd} \tag{8}$$

and therefore

$$\Delta_T F_{cd} = -\epsilon^b \partial_b F_{cd} = \delta_T F_{cd} \ . \tag{9}$$

(c) One has

$$\Delta_T L = \delta_T L = \frac{d}{dx^a}(-\epsilon^a L) \ . \tag{10}$$

Therefore the conserved currents are

$$J^a_{\Delta_T} = \frac{\partial L}{\partial(\partial_a A_c)} \Delta_T A_c + \epsilon^a L = \epsilon^b (F^{ac} F_{bc} - \tfrac{1}{4}\delta^a_b F_{cd} F^{cd}) = T^a_{\ b} \epsilon^b \ . \tag{11}$$

(d) T_{ab} is symmetric:

$$F_{bc} F_a^{\ c} = F_a^{\ c} F_{bc} = F_{ac} F_b^{\ c} \tag{12}$$

(MM^t is symmetric for any matrix M ...). Therefore $T_{ab} = T_{ba}$.

T_{ab} is traceless:

In D spacetime dimensions one has (the expression for T_{ab} is valid for any D)

$$T^a_{\ a} = \eta^{ab} T_{ab} = \eta^{ab} F_{ac} F_b^{\ c} - \tfrac{1}{4}\eta^{ab}\eta_{ab} F_{cd} F^{cd} = F_{ac} F^{ac} - \frac{D}{4} F_{cd} F^{cd} \ , \tag{13}$$

so this is zero precisely for $D = 4$.

(e) From (5) we find

$$\partial^a T_{ab} = (\partial^a F_{ac}) F_b^{\ c} + F_{ac} \partial^a F_b^{\ c} - \tfrac{1}{2}(\partial_b F_{cd}) F^{cd} \ . \tag{14}$$

The Maxwell equations imply that the first term on the right-hand side is zero. In order to be able to combine the remaining terms, we relabel and raise/lower the indices such that

$$F_{ac} \partial^a F_b^{\ c} - \tfrac{1}{2}(\partial_b F_{cd}) F^{cd} = F^{ac} \partial_a F_{bc} - \tfrac{1}{2}(\partial_b F_{ac}) F^{ac} = F^{ac}(\partial_a F_{bc} - \tfrac{1}{2}\partial_b F_{ac}) \ . \tag{15}$$

Since $F^{ac} = -F^{ca}$, only the anti-symmetric part of $\partial_a F_{bc}$ contributes, and therefore we anti-symmetrise explicitly, to find

$$F^{ac}(\partial_a F_{bc} - \tfrac{1}{2}\partial_b F_{ac}) = \tfrac{1}{2} F^{ac}(\partial_a F_{bc} - \partial_c F_{ba} - \partial_b F_{ac}) \tag{16}$$

Finally, by the homogeneous Maxwell equations, the term in brackets is zero,

$$\partial_a F_{bc} - \partial_c F_{ba} - \partial_b F_{ac} = \partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0 \ . \tag{17}$$