# KFT Solutions 04

1. The Homogeneneous Maxwell-Equations

   (a) One has $\partial_a F_{bc} = \partial_a \partial_b A_c - \partial_a \partial_c A_b$ etc. Using the fact that 2nd partial derivatives commute one deduces

   $$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = \partial_a \partial_b A_c - \partial_a \partial_c A_b + \partial_b \partial_c A_a - \partial_b \partial_a A_c + \partial_c \partial_a A_b - \partial_c \partial_b A_a = 0 \tag{1}$$

   (b) $\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$

      i. Two indices equal ($a = b$, say): $\partial_a F_{ac} + \partial_a F_{ca} + \partial_c F_{aa} = 0$ is identically satisfied because $F_{\alpha\alpha} = 0$, $F_{\alpha\gamma} + F_{\gamma\alpha} = 0$.

      ii. All 3 indices spatial, $(a, b, c) = (1, 2, 3)$:

      $$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{21} = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \vec{\nabla}.\vec{B} = 0 \tag{2}$$

      iii. One index time, the others spatial, e.g. $(\alpha, \beta, \gamma) = (0, 1, 2)$:

      $$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = c^{-1}(\partial_t B_3 + \partial_1 E_2 - \partial_2 E_1) = c^{-1}(\vec{\nabla} \times \vec{E} + \partial_t \vec{B})_3 = 0 \tag{3}$$

      (and likewise for the other components).

2. The dual field strength tensor

   The dual field strength tensor is defined by

   $$\tilde{F}^{ab} = \tfrac{1}{2} \epsilon^{abcd} F_{cd} \quad . \tag{4}$$

   Therefore

   $$\begin{aligned} \tilde{F}^{01} &= \tfrac{1}{2} \epsilon^{01cd} F_{cd} = \tfrac{1}{2}(\epsilon^{0123} F_{23} + \epsilon^{0132} F_{32}) = \epsilon^{0123} F_{23} = -F_{23} \\ \tilde{F}^{23} &= \tfrac{1}{2} \epsilon^{23cd} F_{cd} = \epsilon^{2301} F_{01} = \epsilon^{0123} F_{01} = -F_{01} \end{aligned} \tag{5}$$

   etc. In terms of $\vec{E}$ and $\vec{B}$ this means

   $$\tilde{F}^{01} = -B_1 \quad , \quad \tilde{F}^{23} = E_1/c \tag{6}$$

   etc. The equation $\partial_a \tilde{F}^{ab} = 0$ can then be written as

   $$\partial_a \tilde{F}^{a0} = \partial_i \tilde{F}^{i0} = \vec{\nabla} \cdot \vec{B} = 0 \tag{7}$$

   $$\begin{aligned} \partial_a \tilde{F}^{aj} &= \partial_0 \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = -c^{-1} \partial_t B_j - c^{-1} \epsilon_{jik} \partial_i E_k \\ &= -\frac{1}{c} \left( \partial_t \vec{B} + \vec{\nabla} \times \vec{E} \right)_j = 0 \quad , \end{aligned} \tag{8}$$

   which proves the assertion.

3. Lorentz Invariants

Using $\epsilon^{ijl}\epsilon_{ijk} = 2\delta^l_k$, one has

$$I_1 = \tfrac{1}{4}F_{ab}F^{ab} = \tfrac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) = \tfrac{1}{2}\left(\vec{B}^2 - c^{-2}\vec{E}^2\right) \tag{9}$$

$$\begin{aligned} I_2 &= \tfrac{1}{4}F_{ab}\tilde{F}^{ab} = \tfrac{1}{4}\left(F_{0i}\tilde{F}^{0i} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij}\right) \\ &= \tfrac{1}{4}\left(2c^{-1}E_iB_i + \epsilon_{ijk}B_kc^{-1}\epsilon^{ijl}E_l\right) = c^{-1}\vec{E}\cdot\vec{B} \end{aligned} \tag{10}$$

If $\vec{E} = 0$ in one inertial system, then $I_1 > 0$ and $I_2 = 0$ in all inertial systems, and thus $\vec{E}.\vec{B} = 0$ and $|\vec{E}/c| < |\vec{B}|$ in all inertial systems.