Solutions to Assignments 05

1. It is useful to first recall how this works in the case of classical mechanics (i.e. a 0+1 dimensional "field theory"). Consider a Lagrangian $L(q,\dot{q};t)$ that is a total time-derivative, i.e.

$$L(q,\dot{q};t) = \frac{d}{dt}F(q;t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q(t)}\dot{q}(t) \ . \tag{1}$$

Then one has

$$\frac{\partial L}{\partial q(t)} = \frac{\partial^2 F}{\partial q(t)\partial t} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \tag{2}$$

and

$$\frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial F}{\partial q(t)} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} = \frac{\partial^2 F}{\partial t \partial q(t)} + \frac{\partial^2 F}{\partial q(t)^2}\dot{q}(t) \tag{3}$$

Therefore one has

$$\frac{\partial L}{\partial q(t)} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} \qquad \text{identically} \tag{4}$$

Now we consider the field theory case. We define

$$\begin{aligned} L := \frac{d}{dx^a}W^a(\phi;x) &= \frac{\partial W^a}{\partial x^a} + \frac{\partial W^a}{\partial \phi(x)}\frac{\partial \phi(x)}{\partial x^a} \\ &= \partial_a W^a + \partial_\phi W^a \partial_a \phi \end{aligned} \tag{5}$$

to compute the Euler-Lagrange equations for it. One gets

$$\frac{\partial L}{\partial \phi} = \partial_\phi \partial_a W^a + \partial_\phi^2 W^a \partial_a \phi \tag{6}$$

$$\begin{aligned} \frac{d}{dx^b}\frac{\partial L}{\partial(\partial_b \phi)} &= \frac{d}{dx^b}\left(\partial_\phi W^a \delta_a^{\ b}\right) \\ &= \partial_b \partial_\phi W^b + \partial_\phi^2 W^b \partial_b \phi \ . \end{aligned} \tag{7}$$

Thus (since partial derivatives commute) the Euler-Lagrange equations are satisfied identically.

Remark: One can also show the converse: if a Lagrangian L gives rise to Euler-Lagrange equations that are identically satisfied then (locally) the Lagrangian is a total derivative. The proof is simple. Assume that $L(q,\dot{q};t)$ satisfies

$$\frac{\partial L}{\partial q} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \tag{8}$$

identically. The left-hand side does evidently not depend on the acceleration $\ddot{q}$. The right-hand side, on the other hand, will in general depend on $\ddot{q}$ - unless L is at most linear in $\dot{q}$. Thus a necessary condition for L to give rise to identically satisfied Euler-Lagrange equations is that it is of the form

$$L(q,\dot{q};t) = L^0(q;t) + L^1(q;t)\dot{q} \ . \tag{9}$$

Therefore

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}L^1 = \frac{\partial L^1}{\partial t} + \frac{\partial L^1}{\partial q}\dot{q} \tag{10}$$

and

$$\frac{\partial L}{\partial q} = \frac{\partial L^0}{\partial q} + \frac{\partial L^1}{\partial q}\dot{q} \quad . \tag{11}$$

Noting that the 2nd terms of the previous two equations are equal, the Euler-Lagrange equations thus reduce to the condition

$$\frac{\partial L^1}{\partial t} = \frac{\partial L^0}{\partial q} \quad . \tag{12}$$

This means that locally there is a function $F(q;t)$ such that

$$L^0 = \partial_t F \quad , \quad L^1 = \partial_q F \quad , \tag{13}$$

and therefore

$$L = L^0 + L^1\dot{q} = \partial_t F + \partial_q F\dot{q} = \frac{d}{dt}F \quad , \tag{14}$$

as was to be shown. (Proof in the field theory case is analogous)

2. Complex Scalar Field I: Action and Equations of Motion

The action is

$$\begin{aligned} S[\Phi] &= \int d^4x \left(-\tfrac{1}{2}\eta^{ab}\partial_a\Phi\partial_b\Phi^* - W(\Phi,\Phi^*)\right) \\ &= \int d^4x \left(-\tfrac{1}{2}\eta^{ab}\partial_a\phi_1\partial_b\phi_1 - \tfrac{1}{2}\eta^{ab}\partial_a\phi_2\partial_b\phi_2 - V(\phi_1,\phi_2)\right) \end{aligned} \tag{15}$$

(a) Varying ϕ_1 in the 2nd line, while keeping ϕ_2 fixed, one finds

$$\delta S = \int d^4x \left(-\eta^{ab}\partial_a\phi_1\partial_b\delta\phi_1 - (\partial V/\partial\phi_1)\delta\phi_1\right) \quad . \tag{16}$$

Integrating by parts the first term, and dropping the boundary term, one finds the Euler-Lagrange equation $\Box\phi_1 = \partial V/\partial\phi_1$. Analogous for ϕ_2.

(b) Using

$$\begin{aligned} \partial V/\partial\phi_1 &= (\partial W/\partial\Phi)(\partial\Phi/\partial\phi_1) + (\partial W/\partial\Phi^*)(\partial\Phi^*/\partial\phi_1) = (\partial W/\partial\Phi) + (\partial W/\partial\Phi^*) \\ \partial V/\partial\phi_2 &= (\partial W/\partial\Phi)(\partial\Phi/\partial\phi_2) + (\partial W/\partial\Phi^*)(\partial\Phi^*/\partial\phi_2) = i(\partial W/\partial\Phi) - i(\partial W/\partial\Phi^*) \end{aligned} \tag{17}$$

one finds

$$\begin{aligned} \Box\Phi &= \Box\phi_1 + i\Box\phi_2 = \partial V/\partial\phi_1 + i\partial V/\partial\phi_2 = 2\partial W/\partial\Phi^* \\ \Box\Phi^* &= \Box\phi_1 - i\Box\phi_2 = \partial V/\partial\phi_1 - i\partial V/\partial\phi_2 = 2\partial W/\partial\Phi \end{aligned} \tag{18}$$

(c) Varying only Φ^* in the first line of the action, while keeping Φ fixed, one finds

$$\delta S = \int d^4x \left(-\tfrac{1}{2}\eta^{ab}\partial_a\Phi\partial_b\delta\Phi^* - (\partial W/\partial\Phi^*)\delta\Phi^*\right) \quad . \tag{19}$$

Integrating by parts the 1st term, one obtains $(1/2)\Box\Phi$, and thus the correct Euler-Lagrange equation for Φ (analogously for $\Phi \leftrightarrow \Phi^*$).