Solutions to Assignments 06

1. Complex Scalar Field II: Phase Invariance and Noether-Theorem

   (a) If the potential is a function of $\Phi^*\Phi$, both the potential and the derivative terms of the Lagrangian are obviously invariant under

   $$\Phi(x) \to e^{i\theta}\Phi(x) \quad , \quad \Phi^*(x) \to e^{-i\theta}\Phi^*(x) \tag{1}$$

   for constant θ, since in this case the derivatives transform the same way, i.e.

   $$\partial_a\Phi(x) \to e^{i\theta}\partial_a\Phi(x) \quad , \quad \partial_a\Phi^*(x) \to e^{-i\theta}\partial_a\Phi^*(x) \tag{2}$$

   (b) Infinitesimally, one has

   $$\Delta\Phi = i\theta\Phi \quad , \quad \Delta\Phi^* = -i\theta\Phi^* \ , \tag{3}$$

   and therefore the corresponding Noether current is

   $$J^a_\Delta = \frac{\partial L}{\partial(\partial_a\Phi)}\Delta\Phi + \frac{\partial L}{\partial(\partial_a\Phi^*)}\Delta\Phi^* = -(i\theta/2)(\Phi\partial^a\Phi^* - \Phi^*\partial^a\Phi) \tag{4}$$

   where (as usual) $\partial^a = \eta^{ab}\partial_b$. Calculating its divergence, one finds (ignoring the irrelevant constant prefactor, and using the equations of motion)

   $$\begin{aligned}\partial_a(\Phi\partial^a\Phi^* - \Phi^*\partial^a\Phi) &= \partial_a\Phi\partial^a\Phi^* + \Phi\Box\Phi^* - \partial_a\Phi^*\partial^a\Phi - \Phi^*\Box\Phi \\ &= \Phi\Box\Phi^* - \Phi^*\Box\Phi = 2(\Phi\partial W/\partial\Phi - \Phi^*\partial W/\partial\Phi^*)\end{aligned} \tag{5}$$

   This is not (and should not be) zero in general, but it is zero precisely when $W = W(\Phi^*\Phi)$. Indeed, in that case one has

   $$\partial W(\Phi^*\Phi)/\partial\Phi = W'(\Phi^*\Phi)\Phi^* \quad , \quad \partial W(\Phi^*\Phi)/\partial\Phi^* = W'(\Phi^*\Phi)\Phi \ , \tag{6}$$

   and therefore

   $$\Phi\partial W/\partial\Phi - \Phi^*\partial W/\partial\Phi^* = W'(\Phi^*\Phi)\,(\Phi\Phi^* - \Phi^*\Phi) = 0 \ . \tag{7}$$

2. Complex Scalar Field III: Gauge Invariance and Minimal Coupling

   (a) Under

   $$\Phi(x) \to e^{i\theta(x)}\Phi(x) \quad , \quad \Phi^*(x) \to e^{-i\theta(x)}\Phi^*(x) \quad , \quad A_a(x) \to A_a(x) + \partial_a\theta(x) \tag{8}$$

   the partial derivative transforms as

   $$\partial_a\Phi \to \partial_a(e^{i\theta}\Phi) = e^{i\theta}(\partial_a\Phi + i(\partial_a\theta)\Phi) \tag{9}$$

Therefore the covariant derivative

$$D_a\Phi = \partial_a\Phi - iA_a\Phi \quad , \quad D_a\Phi^* = \partial_a\Phi^* + iA_a\Phi^* \quad . \tag{10}$$

transforms as

$$\begin{aligned} D_a\Phi &\to \mathrm{e}^{i\theta}(\partial_a\Phi + i(\partial_a\theta)\Phi) - i\mathrm{e}^{i\theta}A_a\Phi - i\mathrm{e}^{i\theta}(\partial_a\theta)\Phi \\ &= \mathrm{e}^{i\theta}(\partial_a\Phi - iA_a\Phi) = \mathrm{e}^{i\theta}D_a\Phi \end{aligned} \tag{11}$$

Likewise

$$D_a\Phi^* \to \mathrm{e}^{-i\theta}D_a\Phi^* \quad . \tag{12}$$

(b) It is now obvious that the action

$$S[\Phi, A] = \int d^4x \left(-\tfrac{1}{2}\eta^{ab}D_a\Phi D_b\Phi^* - W(\Phi\Phi^*)\right) \tag{13}$$

ist gauge invariant.

(c) The action is

$$S = S_{\mathrm{Maxwell}}[A] + S[\Phi, A] = \int d^4x(-\tfrac{1}{4}F^2) + S[\Phi, A] \quad . \tag{14}$$

The equations of motion for Φ and Φ^* are simply the covariant versions of the equations of motion from Exercise 05.2, namely

$$D^aD_a\Phi = 2\partial W/\partial\Phi^* \quad , \quad D^aD_a\Phi^* = 2\partial W/\partial\Phi \quad . \tag{15}$$

Variation with respect to A leads to

$$\delta S = \int d^4x \left(\partial_a F^{ab} + J^b\right)\delta A_b \tag{16}$$

where

$$J^b = (i/2)\left(\Phi D^b\Phi^* - \Phi^* D^b\Phi\right) \tag{17}$$

The equations of motion $\partial_a F^{ab} + J^b = 0$ imply (and therefore require) that $\partial_b J^b = 0$. Let us show that this equation is satisfied as a consequence of the equations of motion for Φ.

First of all, we have

$$\partial_b(\Phi D^b\Phi^*) = \partial_b\Phi D^b\Phi^* + \Phi\partial_b D^b\Phi^* \quad . \tag{18}$$

Adding and subtracting $+iA_b\Phi$, we can write this as

$$\partial_b(\Phi D^b\Phi^*) = D_b\Phi D^b\Phi^* + \Phi D_b D^b\Phi^* \quad . \tag{19}$$

Since the first term is invariant under the exchange $\Phi \leftrightarrow \Phi^*$, one finds

$$\partial_b\left(\Phi D^b\Phi^* - \Phi^* D^b\Phi\right) = \Phi D_b D^b\Phi^* - \Phi^* D_b D^b\Phi \tag{20}$$

Note that this is just the covariant version of the divergence of the Noether current in Exercise 1, and the remaining step in the proof that this vanishes for a solution to the equations of motion is now identical to that in Exercise 1.