

SOLUTIONS 07

1. Noether Energy-Momentum Tensor

The claim follows from

$$\begin{aligned}
 \frac{d}{dx^a} \Theta^a_b &= \frac{d}{dx^a} \left(-\frac{\partial L}{\partial(\partial_a \Phi^A)} \partial_b \Phi^A + \delta^a_b L \right) \\
 &= -\left(\frac{d}{dx^a} \frac{\partial L}{\partial(\partial_a \Phi^A)} \right) \partial_b \Phi^A - \frac{\partial L}{\partial(\partial_a \Phi^A)} \partial_a \partial_b \Phi^A + \frac{d}{dx^b} L \\
 &= \left(\frac{\partial L}{\partial \Phi^A} - \frac{d}{dx^a} \frac{\partial L}{\partial(\partial_a \Phi^A)} \right) \partial_b \Phi^A + \frac{\partial}{\partial x^b} L
 \end{aligned} \tag{1}$$

2. Noether Energy-Momentum Tensor for a Scalar Field

The action is

$$\begin{aligned}
 S[\phi] &= \int d^4x \left(-\frac{1}{2} \eta^{ab} \partial_a \phi \partial_b \phi - V(\phi) \right) \equiv \int d^4x \left(-\frac{1}{2} (\partial\phi)^2 - V(\phi) \right) \\
 \Rightarrow \delta S[\phi] &= \int d^4x \left(-\eta^{ab} \partial_a \phi \partial_b \delta\phi - V'(\phi) \delta\phi \right) \\
 &= \int d^4x \left(\eta^{ab} \partial_a \partial_b \phi - V'(\phi) \right) \delta\phi = \int d^4x (\square\phi - V'(\phi)) \delta\phi \\
 \Rightarrow \square\phi &= V'(\phi)
 \end{aligned} \tag{2}$$

The energy-momentum tensor is

$$\Theta_{ab} = -\frac{\partial L}{\partial(\partial^a \phi)} \partial_b \phi + \eta_{ab} L = \partial_a \phi \partial_b \phi - \eta_{ab} \left(\frac{1}{2} (\partial\phi)^2 + V(\phi) \right) . \tag{3}$$

The claim follows from

$$\begin{aligned}
 \partial^a \Theta_{ab} &= (\square\phi) \partial_b \phi + \partial_a \phi \partial^a \partial_b \phi - \partial_b \left(\frac{1}{2} (\partial\phi)^2 + V(\phi) \right) \\
 &= (\square\phi) \partial_b \phi + \partial_a \phi \partial^a \partial_b \phi - \partial_b \partial^a \phi \partial_a \phi - V'(\phi) \partial_b \phi \\
 &= (\square\phi - V'(\phi)) \partial_b \phi .
 \end{aligned} \tag{4}$$

3. Maxwell Energy-Momentum Tensor

$$T_{ab} = F_{ac} F_b^c - \frac{1}{4} \eta_{ab} F_{cd} F^{cd} . \tag{5}$$

(a) see the Lecture Notes

(b)

$$\Delta_T A_c = \delta_T A_c + \partial_c (\epsilon^b A_b) = -\epsilon^b F_{bc} , \tag{6}$$

is a variation, and therefore

$$\Delta_T F_{cd} = \partial_c \Delta_T A_d - \partial_d \Delta_T A_c = -\epsilon^b (\partial_c F_{bd} - \partial_d F_{bc}) . \tag{7}$$

The Bianchi identity implies

$$\partial_c F_{bd} - \partial_d F_{bc} = -\partial_c F_{db} - \partial_d F_{bc} = +\partial_b F_{cd} \quad (8)$$

and therefore

$$\Delta_T F_{cd} = -\epsilon^b \partial_b F_{cd} = \delta_T F_{cd} \quad . \quad (9)$$

(c) One has

$$\Delta_T L = \delta_T L = \frac{d}{dx^a} (-\epsilon^a L) \quad . \quad (10)$$

Therefore the conserved currents are

$$J_{\Delta_T}^a = \frac{\partial L}{\partial(\partial_a A_c)} \Delta_T A_c + \epsilon^a L = \epsilon^b (F^{ac} F_{bc} - \frac{1}{4} \delta_b^a F_{cd} F^{cd}) = T_b^a \epsilon^b \quad . \quad (11)$$

(d) T_{ab} is symmetric:

$$F_{bc} F_a^c = F_a^c F_{bc} = F_{ac} F_b^c \quad (12)$$

(MM^t is symmetric for any matrix $M \dots$). Therefore $T_{ab} = T_{ba}$.

T_{ab} is traceless:

In D spacetime dimensions one has (the expression for T_{ab} is valid for any D)

$$T_a^a = \eta^{ab} T_{ab} = \eta^{ab} F_{ac} F_b^c - \frac{1}{4} \eta^{ab} \eta_{ab} F_{cd} F^{cd} = F_{ac} F^{ac} - \frac{D}{4} F_{cd} F^{cd} \quad , \quad (13)$$

so this is zero precisely for $D = 4$.

(e) From (5) we find

$$\partial^a T_{ab} = (\partial^a F_{ac}) F_b^c + F_{ac} \partial^a F_b^c - \frac{1}{2} (\partial_b F_{cd}) F^{cd} \quad . \quad (14)$$

The Maxwell equations imply that the first term on the right-hand side is zero. In order to be able to combine the remaining terms, we relabel and raise/lower the indices such that

$$F_{ac} \partial^a F_b^c - \frac{1}{2} (\partial_b F_{cd}) F^{cd} = F^{ac} \partial_a F_{bc} - \frac{1}{2} (\partial_b F_{ac}) F^{ac} = F^{ac} (\partial_a F_{bc} - \frac{1}{2} \partial_b F_{ac}) \quad . \quad (15)$$

Since $F^{ac} = -F^{ca}$, only the anti-symmetric part of $\partial_a F_{bc}$ contributes, and therefore we anti-symmetrise explicitly, to find

$$F^{ac} (\partial_a F_{bc} - \frac{1}{2} \partial_b F_{ac}) = \frac{1}{2} F^{ac} (\partial_a F_{bc} - \partial_c F_{ba} - \partial_b F_{ac}) \quad (16)$$

Finally, by the homogeneous Maxwell equations, the term in brackets is zero,

$$\partial_a F_{bc} - \partial_c F_{ba} - \partial_b F_{ac} = \partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0 \quad . \quad (17)$$