Solutions to Assignments 01

1. The Lorentz Group

(a) The first claim follows from multiplicativity of the determinant (and invariance under transposition):

$$L^T\eta L = \eta \quad \Rightarrow \quad \det(L^T\eta L) = \det(\eta) \quad \Rightarrow \quad \det(L)^2 = +1 \ . \tag{1}$$

The second claim follows fom writing $(L^T\eta L)_{00} = \eta_{00}$ explicitly,

$$\begin{aligned} & \eta_{ab}L^a{}_0 L^b{}_0 \stackrel{!}{=} \eta_{00} = -1 \\ \Rightarrow \quad & \eta_{00}L^0_0 L^0{}_0 + \eta_{ik}L^i_0 L^k{}_0 = -(L^0_0)^2 + \delta_{ik}L^i_0 L^k{}_0 = -1 \\ \Rightarrow \quad & (L^0_0)^2 = 1 + \delta_{ik}L^i_0 L^k{}_0 \geq 1 \ . \end{aligned} \tag{2}$$

(b) It is trivial to verify that

$$L_1^T\eta L_1 = \eta \quad , \quad L_2^T\eta L_2 = \eta \quad \Rightarrow \quad (L_1L_2)^T\eta(L_1L_2) = \eta \ . \tag{3}$$

Existence of an inverse L^{-1} follows from $\det L \neq 0$ (shown above). That $L \in \mathcal{L} \Rightarrow L^{-1} \in \mathcal{L}$ follows from

$$L^T\eta L = \eta \quad \Leftrightarrow \quad \eta = (L^{-1})^T\eta L^{-1} \ . \tag{4}$$

Thus Lorentz transformations indeed form a group.

(c)
- Matrix Calculation

  One can use the good old matrix formalism. For an infinitesimal Lorentz transformation, with $L = \mathbb{1}+\omega$, the defining condition $L^T\eta L = \eta$ reduces to

$$\begin{aligned} (\mathbb{1}+\omega)^T\eta(\mathbb{1}+\omega) = \eta \quad & \Rightarrow \quad \eta + \omega^T\eta + \eta\omega = \eta \\ & \Rightarrow \quad (\eta\omega)^T + (\eta\omega) = 0 \ . \end{aligned} \tag{5}$$

  In components, $\eta\omega$ is the matrix

$$(\eta\omega)_{ab} = \eta_{ac}\omega^c{}_b \equiv \omega_{ab} \tag{6}$$

  and thus anti-symmetry of $\eta\omega$ means $\omega_{ba} = -\omega_{ab}$.
- Calculation in Components

  For the following it will be much more useful to learn how to do such calculations directly in components. With $L^a_b = \delta^a_b + \omega^a_b$, one has (writing out everything in detail, please make sure that you understand all the steps)

$$\begin{aligned}\eta_{ab}L^a{}_cL^b{}_d &= \eta_{ab}(\delta^a_c+\omega^a{}_c)(\delta^b_d+\omega^b{}_d)\\ &= \eta_{ab}\delta^a_c\delta^b_d+\eta_{ab}\delta^a_c\omega^b{}_d+\eta_{ab}\omega^a{}_c\delta^b_d\\ &= \eta_{cd}+\eta_{cb}\omega^b{}_d+\eta_{ad}\omega^a{}_c=\eta_{cd}+\eta_{cb}\omega^b{}_d+\eta_{da}\omega^a{}_c\\ &= \eta_{cd}+(\eta\omega)_{cd}+(\eta\omega)_{dc}\stackrel{!}{=}\eta_{cd}\end{aligned} \tag{7}$$

which implies the anti-symmetry of $\eta\omega$,

$$(\eta\omega)_{cd}=-(\eta\omega)_{dc}\ . \tag{8}$$

Example: in $(1+1)$ dimensions, a boost with rapidity α has the matrix form

$$L(\alpha)=\begin{pmatrix}\cosh\alpha & -\sinh\alpha\\ -\sinh\alpha & \cosh\alpha\end{pmatrix} \tag{9}$$

For small (infinitesimal) rapidities α, $L(\alpha)$ reduces to

$$L(\alpha)\approx\begin{pmatrix}1&0\\0&1\end{pmatrix}+\alpha\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\equiv\mathbb{1}+\omega\ . \tag{10}$$

Note that the second term is not (yet) anti-symmetric, but in accordance with the general result derived above, its product with η is,

$$\eta\omega=\alpha\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\-1&0\end{pmatrix}=\begin{pmatrix}0&+\alpha\\-\alpha&0\end{pmatrix}\ . \tag{11}$$