Solutions to Assignments 02

1. Tensor Algebra: Lorentz Tensors

   By definition Lorentz vectors and covectors transforms as

$$\bar{v}^a = L^a_{\ b} v^b \quad , \quad \bar{u}_a = \Lambda_a^{\ b} u_b \quad \text{with} \quad \Lambda_a^{\ b} L^a_{\ c} = \delta^b_{\ c} \ . \tag{1}$$

   This definition is such that the *contraction* between a vector and a covector is a *scalar* (invariant under Lorentz transformations),

$$\bar{u}_a \bar{v}^a = \Lambda_a{}^b L^a{}_c u_b v^c = \delta^b{}_c u_b v^c = u_b v^b = u_a v^a \ . \tag{2}$$

   Higher rank tensors transform like products of vectors and covectors, i.e. a (p,q) tensor transforms with p factors of L and q factors of Λ and is written as an object with p upper indices and q lower indices.

   By the same calculation as above one then finds that any contracted pair of indices on a tensor (summation over one "upper" and one "lower" index) is invariant. Therefore the tensor type of the resulting object can be read off just by looking at the number of uncontracted upper and lower indices. For example:

   (a) for the contraction of a $(2,0)$-tensor and a $(0,1)$-tensor (covector) one has

$$\bar{T}^{ab} \bar{u}_b = L^a_{\ c} L^b_{\ d} \, T^{cd} \, \Lambda_b{}^e u_e = L^a_{\ c} \delta^e_{\ d} \, T^{cd} \, u_e = L^a_{\ c} \, T^{cd} \, u_d \tag{3}$$

   so that $T^{ab} u_b$ transforms like (and therefore is) a $(1,0)$-tensor (vector), as indicated by the fact that this object has one free upper index.

   (b) likewise the trace of a $(1,1)$-tensor is a scalar,

$$\bar{T}^a_{\ a} = L^a_{\ b} \Lambda_a^{\ c} T^b_{\ c} = \delta^c{}_b T^b{}_c = T^b{}_b = T^a{}_a \ . \tag{4}$$

   Note: the trace of a $(0,2)$-tensor T_{ab} is not well-defined without using the Minkowski metric, i.e. something like

$$\mathsf{trace}(T_{ab}) \stackrel{?}{=} \sum_\alpha T_{aa} \qquad (???) \tag{5}$$

   is not Lorentz-invariant and therefore depends on the inertial system in which it is evaluated, and is therefore unlikely to be a useful quantity. However, with the help of the Minkowski metric one can define a Lorentz-invariant trace (i.e. a scalar) via

$$T_{ab} \to T^a_{\ b} = \eta^{ac} T_{cb} \to T^a_{\ a} = \eta^{ac} T_{ca} \qquad \checkmark \tag{6}$$

   ("taking the trace with respect to η"). This is now manifestly a scalar.

2. Tensor Analysis: Lorentz Tensors and their Derivatives

As recalled in the previous exercise, the formalism is designed in such a way that the transformation behaviour (tensorial nature) can just be read off from the free indices. This extends to partial derivatives of tensors.

(a) Set

$$\frac{\partial}{\partial \bar{x}^a} = M_a{}^b \frac{\partial}{\partial x^b} \tag{7}$$

for some matrix M to be determined. We will show that $M = \Lambda$. To that end, use the chain rule to write

$$\bar{x}^a = L^a_{\ b} x^b \quad \Rightarrow \quad \frac{\partial}{\partial x^b} = \frac{\partial \bar{x}^c}{\partial x^b}\frac{\partial}{\partial \bar{x}^c} = L^c_{\ b}\frac{\partial}{\partial \bar{x}^c} \ , \tag{8}$$

and plug this into the previous equation, to conclude

$$\frac{\partial}{\partial \bar{x}^a} = M_a{}^b L^c_{\ b}\frac{\partial}{\partial \bar{x}^c} \quad \Rightarrow \quad M_a{}^b L^c_{\ b} = \delta^a{}_c \quad \Rightarrow \quad M_a{}^b = \Lambda_a{}^b \ . \tag{9}$$

Note: Just in case this was too quick for you, and you want to object that the equation

$$M_a{}^b L^c_{\ b} = \delta^a{}_c \tag{10}$$

we just obtained for M is not the same as the equation

$$\Lambda_a{}^b L^a_{\ c} = \delta^b_{\ c} \ . \tag{11}$$

characterising and defining Λ in terms of L, I will add: yes, they are not the same but they are equivalent. The first one says $ML^T = \mathbb{1}$, and the second says $\Lambda^T L = \mathbb{1}$, but both have the same solution

$$\begin{aligned} \Lambda^T L = \mathbb{1} \quad &\Rightarrow \quad \Lambda^T = L^{-1} \quad \Rightarrow \Lambda = (L^T)^{-1} \\ ML^T = \mathbb{1} \quad &\Rightarrow \quad M = (L^T)^{-1} = \Lambda \end{aligned} \tag{12}$$

(b) It follows from the above (i.e. from what is above the linear algebra trivialities) that the partial derivative of a scalar function f, i.e. $\bar{f}(\bar{x}) = f(x)$, is a covector, and we will abbreviate it by $\partial_a f$ etc.

More generally, ∂_a acting on a (p,q)-tensor field gives a $(p,q+1)$-tensor field. Then the answers are immediately

$$V^a \partial_a f \quad \text{scalar} \quad , \quad V^a \partial_b f \quad \text{(1,1)-tensor} \quad , \quad \partial_a V^a \quad \text{scalar}$$
$$f \partial_a V^a \quad \text{scalar} \quad , \quad \partial_a U_b \quad \text{(0,2)-tensor}$$
$$\partial_a \partial_b f \quad \text{(0,2)-tensor} \quad , \quad V^a \partial_b U_a \quad \text{(0,1)-tensor or covector}$$