# KFT Solutions 03

1. Particle with constant acceleration

   We consider the (1+1)-dimensional worldline

$$x^a(\lambda) = (g^{-1}\sinh g\lambda, g^{-1}\cosh g\lambda) \tag{1}$$

   (a) One has $(x^1)^2 - (x^0)^2 = g^{-2}$. This is a hyperbola. Moreover, manifestly along this worldline $x^1 > |x^0|$, so this is a hyperbola in the right-hand quadrant of Minkowski space. By inspection / drawing a picture, one sees that this curve is timelike.

   (b) The tangent vector to the curve has components

$$\frac{dx^a}{d\lambda} = (\cosh g\lambda, \sinh g\lambda) \ . \tag{2}$$

   Therefore its Minkowski norm is

$$\eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = -\cosh^2 g\lambda + \sinh^2 g\lambda = -1 < 0 \ . \tag{3}$$

   This confirms that the worldline is indeed timelike everywhere.

   (c) By definition, a worldline parametrised by proper time $\tau$, with corresponding 4-velocity $u^a$, satisfies (in units where $c = 1$)

$$\eta_{ab}u^a u^b \equiv u^a u_a = -1 \ . \tag{4}$$

   On the other hand, as we saw above, the velocity with respect to $\lambda$ also satisfies this relation everywhere along the worldline. It follows that $\lambda = \tau$ up to an additive constant.

   (d) Setting $\lambda = \tau$, the 4-velocity has components

$$u^a = (\cosh g\tau, \sinh g\tau) \tag{5}$$

   and the 4-acceleration has components

$$a^a = (g\sinh g\tau, g\cosh g\tau) \ . \tag{6}$$

   (e) The Minkowski-norm of the acceleration is

$$a^a a_a = \eta_{ab}a^a a^b = g^2(\cosh^2 g\tau - \sinh^2 g\tau) = g^2 \tag{7}$$

   which is constant. Thus this worldline describes a particle with constant acceleration $g$.

(f) For $c \neq 1$, we have $u^a u_a = -c^2$. Moreover, the arguments of the hyperbolic functions need to be dimensionless (it makes no sense to take a non-linear function of a dimensionful variable!). If $g$ has the dimension of an acceleration, and the proper time $\tau$ has the dimension of time, then $g\tau$ has the dimension of a velocity. Thus $g\tau/c$ is dimensionless. This determines that

$$\begin{aligned} x^a(\tau) &= (c^2/g)(\sinh g\tau/c, \cosh g\tau/c) \\ u^a(\tau) &= c(\cosh g\tau/c, \sinh g\tau/c) \qquad (\Rightarrow u^a u_a = -c^2) \\ a^a(\tau) &= g(\sinh g\tau/c, \cosh g\tau/c) \qquad (\Rightarrow a^a a_a = +g^2) \end{aligned} \tag{8}$$

(g) We first determine the coordinate velocity $v = dx/dt$ as a function of $\tau$,

$$v(\tau) = \frac{dx}{dt}(\tau) = \frac{\dot{x}(\tau)}{\dot{t}(\tau)} = \frac{c \sinh g\tau/c}{\cosh g\tau/c} = c \tanh g\tau/c \ . \tag{9}$$

To write this as a function of $t$ we note that

$$\sinh g\tau/c = gt/c \quad \Rightarrow \quad \cosh g\tau/c = (1 + g^2t^2/c^2)^{1/2} \ , \tag{10}$$

so that

$$v(t) = c\frac{gt/c}{(1 + g^2t^2/c^2)^{1/2}} = \frac{gt}{(1 + g^2t^2/c^2)^{1/2}} \ . \tag{11}$$

(h) For a Newtonian particle with acceleration $g$ one would have had $v_N(t) = gt$. Thus the difference between the Newtonian and special relativistic results is encoded in the factor $(1 + g^2t^2/c^2)^{-1/2}$. In particular,

- the Newtonian particle would have reached the speed of light at the time $t_c = c/g$, i.e. $v_N(t_c) = c$, at which the special-relativistic particle has the speed

$$v(t_c) = c/\sqrt{2} \ , \tag{12}$$

- and for $t \to \infty$ one has $v(t) \to c$.