KFT Solutions 04

1. Action for a free particle

The action is

$$S[x] = -mc^2 \int d\tau = -mc^2 \int d\lambda (d\tau/d\lambda) \equiv \int d\lambda\ L_\lambda \tag{1}$$

with $L_\lambda = -mc^2(d\tau/d\lambda) = -mc(-\eta_{ab}x'^a x'^b)^{1/2}$.

(a) In order to determine the momentum

$$p_a = \frac{\partial L_\lambda}{\partial x'^a} \tag{2}$$

conjugate to $x^a(\lambda)$, and to avoid index confusions, let us write the Lagrangian (without using the index ^{a}) as

$$L_\lambda = -mc(-\eta_{cb}x'^c x'^b)^{1/2} \quad . \tag{3}$$

Using the fact that

$$\frac{\partial x'^c}{\partial x'^a} = \delta^c_a \tag{4}$$

and that η_{cb} is symmetric, one has

$$\frac{\partial}{\partial x'^a}\left(\eta_{cb}x'^c x'^b\right) = 2\eta_{ab}x'^b \quad . \tag{5}$$

Therefore

$$\begin{aligned} p_a = \frac{\partial L_\lambda}{\partial x'^a} &= -mc\ \tfrac{1}{2}\left(-\eta_{ab}x'^a x'^b\right)^{-1/2}\left(-2\eta_{ab}x'^b\right) \\ &= mc\ \eta_{ab}\left(\frac{cd\tau}{d\lambda}\right)^{-1}\frac{dx^b}{d\lambda} = m\eta_{ab}\frac{dx^b}{d\tau} = mu_a \end{aligned} \tag{6}$$

or $p^a = mu^a$. In an inertial system with cordinates $(x^0 = ct, x^k)$, and with $v^k = dx^k/dt$ one has

$$p^0 = m\gamma(v)c = E/c \quad , \quad p^k = m\gamma(v)v^k \quad . \tag{7}$$

(b) See the KFT Lecture Notes, section 4.4 for the detailed argument.

(c) The Euler-Lagrange equation is

$$\frac{d}{d\lambda}\frac{\partial L_\lambda}{\partial x'^a} - \frac{\partial L_\lambda}{\partial x^a} = \frac{d}{d\lambda}\frac{\partial L_\lambda}{\partial x'^a} = 0 \tag{8}$$

and

$$\frac{d}{d\lambda}\frac{\partial L_\lambda}{\partial x'^a} = \frac{d\tau}{d\lambda}\frac{d}{d\tau}mu_a = \left(m\eta_{ab}\frac{d\tau}{d\lambda}\right)\frac{d^2x^b}{d\tau^2} = 0 \quad \Leftrightarrow \quad \frac{d^2x^b}{d\tau^2} = 0 \quad . \tag{9}$$

(d) The Lagrangian is

$$L_t = -mc^2 \frac{d\tau}{dt} = -mc^2 \sqrt{1 - \vec{v}^2/c^2} = -mc^2 \gamma(v)^{-1} \quad . \tag{10}$$

Thus the canonical momenta are

$$p_k^{(c)} = \frac{\partial L_t}{\partial v^k} = (-mc^2)\gamma(v)(-v_k/c^2) = m\gamma(v)v_k = p_k \quad . \tag{11}$$

Note: Here $v_k = \delta_{kl} v^l$, which arises from

$$v^2 = \delta_{kl} v^k v^l \equiv v_k v^k \quad \Rightarrow \quad \frac{\partial}{\partial v^k} v^2 = 2v_k \quad . \tag{12}$$

If one wants to be notationally correct, then one should write things in this way. However, for spatial Euclidean indices it is not always strictly necessary to distinguish between covariant and contravariant components, since numerically they are equal, $v_k = v^k$, and $p_k = p^k$.

The canonical Hamiltonian is

$$\begin{aligned} H = p_k^{(c)} v^k - L_t &= m\gamma(v)\vec{v}^2 + mc^2\gamma(v)^{-1} \\ &= m\gamma(v)(\vec{v}^2 + c^2(1 - \vec{v}^2/c^2)) = m\gamma(v)c^2 = E = cp^0 \quad . \end{aligned} \tag{13}$$

Remark:

The above derivation of the relativistic spatial momenta p_k from the Lagrangian L_t can be reverse-engineered to provide a poor man's "derivation" of the action $S = -mc^2 \int d\tau$ from the knowledge of the relativistic momenta $p^k = m\gamma(v)v^k$.

Namely, assume that you are in the inertial system with coordinates (t, x^k) and that you are looking for a standard type Lagrangian

$$L = L(x^k, v^k; t) \tag{14}$$

where $v^k = dx^k/dt$ are the ordinary Newtonian coordinate velocities, and where initially we allow a possible explicit dependence of the Lagrangian L on time t. Then the condition that the canonical momenta conjugate to the ordinary velocities v^k are the relativistic momenta p_k implies

$$\frac{\partial L}{\partial v^k} \stackrel{!}{=} p_k = m\gamma(v)v_k \quad \Rightarrow \quad L = -mc^2\gamma(v)^{-1} + f(x^k, t) \quad , \tag{15}$$

where $f(x^k, t)$ is some undetermined function of the coordinates x^k and t. Now translation invariance in x^k and t implies that f is just some undetermined constant (which we will set to zero). Then

$$f(x^k, t) = 0 \quad \Rightarrow \quad L = L_t = -mc^2\gamma(v)^{-1} \quad , \tag{16}$$

and therefore

$$S = \int L_t dt = -mc^2 \int dt \gamma(v)^{-1} = -mc^2 \int d\tau \quad . \tag{17}$$

2. Noether-Theorem I: General

(a) Using $\delta x'^a(\lambda) = (d/d\lambda)\delta x^a$, one has

$$\delta L = \frac{\partial L}{\partial x^a}\delta x^a + \frac{\partial L}{\partial x'^a}\delta x'^a = \frac{\partial L}{\partial x^a}\delta x^a + \frac{\partial L}{\partial x'^a}\frac{d}{d\lambda}\delta x^a$$
$$= \left(\frac{\partial L}{\partial x^a} - \frac{d}{d\lambda}\frac{\partial L}{\partial x'^a}\right)\delta x^a + \frac{d}{d\lambda}\left(\frac{\partial L}{\partial x'^a}\delta x^a\right) \equiv \mathcal{E}_a \delta x^a + \frac{d}{d\lambda}(p_a \delta x^a) \ . \quad (18)$$

(b) The above identity (the VME) implies that

$$\delta_s L = 0 \quad \Rightarrow \quad \frac{d}{d\lambda}(p_a \delta_s x^a) = 0 \quad (19)$$

for any x^a satisfying the Euler-Lagrange equations $\mathcal{E}_a = 0$.

3. Noether-Theorem II: Free Relativistic Particle

(a) Since $L_\lambda = -mc(-\eta_{ab}x'^a x'^b)^{1/2}$ does not depend on $x^a(\lambda)$, and by the result of Exercise 03.2a, for any variation δx^a one has

$$\delta L_\lambda = p_a \delta x'^a \quad (20)$$

with $p^a = mu^a$ the 4-momentum, related to x'^a by

$$x'^a = (1/m)(d\tau/\delta\lambda)p^a \equiv \mu p^a \ . \quad (21)$$

For an infinitesimal Poincaré transformation one has

$$\delta_s x^a = \epsilon^a + \omega^a_{\ b} x^b \quad \Rightarrow \quad \delta_s x'^a = \omega^a_{\ b} x'^b = \mu \omega^a_{\ b} p^b \ . \quad (22)$$

Thus

$$\delta_s L_\lambda = \mu p_a \omega^a_{\ b} p^b = \mu \omega_{ab} p^a p^b = 0 \quad (23)$$

because $p^a p^b = p^b p^a$ is symmetric, and $\omega_{ab} = -\omega_{ba}$ is anti-symmetric.

(b) For translations one has

$$\delta_s x^a = \epsilon^a \quad \Rightarrow \quad p_a \delta_s x^a = \epsilon^a p_a \quad (24)$$

Because ϵ^a is an arbitrary constant vector, the conserved quantity is the momentum p_a (which unifies spatial momentum and energy). For Lorentz transformations one has

$$\delta_s x^a = \omega^a_{\ b} x^b \quad \Rightarrow \quad p_a \delta_s x^a = p_a \omega^a_{\ b} x^b = \omega_{ab} p^a x^b = \tfrac{1}{2}\omega_{ab}(p^a x^b - p^b x^a) \ . \quad (25)$$

Because ω_{ab} is an arbitrary anti-symmetric constant matrix, the conserved quantity is the anti-symmetric combination

$$\mathcal{L}^{ab} = p^a x^b - p^b x^a \quad (26)$$

(which unifies angular momentum $\mathcal{L}^{ik}$ and the conserved quantity $\mathcal{L}^{0k}$ associated to Lorentzian boosts).