KFT Solutions 05

1. Inhomogeneous Maxwell-Equations and Potentials

(a) With $A_a = (-\phi/c, \vec{A})$, the transformation $A_a \to A_a + \partial_a \Psi$ translates into

$$A_0 = -\phi/c \to -\phi/c + \partial_0 \Psi = -\phi/c + \partial_t \Psi/c \quad \Leftrightarrow \quad \phi \to \phi - \partial_t \Psi \tag{1}$$

and $A_i \to A_i + \partial_i \Psi \Leftrightarrow \vec{A} \to \vec{A} + \vec{\nabla}\Psi$.

(b) Under the gauge transformation $A_b \to A_b + \partial_b \Psi$, F_{ab} transforms as

$$\partial_a A_b - \partial_b A_a \to \partial_a A_b + \partial_a \partial_b \Psi - \partial_b A_a - \partial_b \partial_a \Psi = \partial_a A_b - \partial_b A_a \tag{2}$$

and therefore F_{ab} is gauge-invariant.

(c) With $A_a = (-\phi/c, \vec{A})$ one has

$$\begin{aligned} F_{0k} = -F_{k0} &= \partial_0 A_k - \partial_k A_0 = c^{-1}(\partial_t A_k + \partial_k \phi) = -E_k/c \\ F_{ik} &= \partial_i A_k - \partial_k A_i = \epsilon_{ik\ell} B_\ell \quad (F_{12} = B_3 \quad \text{etc.}) \end{aligned} \tag{3}$$

and therefore, with $F^{ab} = \eta^{ac}\eta^{bd} F_{cd}$,

$$F^{0k} = -F^{k0} = -F_{0k} = E_k/c \quad , \quad F^{ik} = F_{ik} = \epsilon_{ik\ell} B_\ell \ . \tag{4}$$

(d) Thus, with $J^a = \mu_0(\rho c, \vec{J})$ one has

$$\partial_a F^{a0} = \partial_k F^{k0} = -c^{-1} \vec{\nabla}.\vec{E} = -\rho/(\epsilon_0 c) = -\mu_0 c \rho = -J^0 \tag{5}$$

and

$$\begin{aligned} \partial_a F^{a1} &= \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = c^{-2} \partial_t E_1 - \partial_2 B_3 + \partial_3 B_2 \\ &= -(\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E})_1 = -J_1 = -J^1 \end{aligned} \tag{6}$$

(and likewise for the 2- and 3-components).

(e) One has

$$\partial_a F^{ab} = \partial_a \partial^a A^b - \partial_a \partial^b A^a = \Box A^b - \partial^b \partial_a A^a = -J^b \tag{7}$$

and therefore $\Box A_b - \partial_b \partial_a A^a = -J_b$.

(f) From $\partial_a F^{ab} = -J^b$ one deduces $-\partial_b J^b = \partial_b \partial_a F^{ab} = 0$ because $F^{ab} = -F^{ba}$ is anti-symmetric while $\partial_a \partial_b = \partial_b \partial_a$ is symmetric.