KFT Solutions 06

1. The Homogeneneous Maxwell-Equations

   (a) For $G_{ab} = G_{[ab]}$ (totally) anti-symmetric, the definition

   $$H_{abc} = \partial_a G_{bc} + \partial_b G_{ca} + \partial_c G_{ab} \tag{1}$$

   implies that when one exchanges the two indices a, b, one gets

   $$H_{bac} = \partial_b G_{ac} + \partial_a G_{cb} + \partial_c G_{ba} = -\partial_b G_{ca} - \partial_a G_{bc} - \partial_c G_{ab} = -H_{abc} \tag{2}$$

   (and likewise for the index pairs a, c and b, c).

   (b) One has

   $$\partial_a F_{bc} = \partial_a \partial_b A_c - \partial_a \partial_c A_b \tag{3}$$

   etc. Because 2nd partial derivatives commute, the 1st term is symmetric in a, b, and the 2nd term is symmetric in a, c. Therefore neither of them can appear in the totally anti-symmetric combination $\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}$. Thus this linear combination is identically zero,

   $$F_{ab} = \partial_a A_b - \partial_b A_a \quad \Rightarrow \quad \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} \equiv 0 \quad \text{(Bianchi Identity)} \ . \tag{4}$$

   One can of course also verify this explicitly,

   $$\partial_a \partial_b A_c - \partial_a \partial_c A_b + \partial_b \partial_c A_a - \partial_b \partial_a A_c + \partial_c \partial_a A_b - \partial_c \partial_b A_a = 0 \ . \tag{5}$$

   (c) We consider the equations

   $$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0 \tag{6}$$

   with F_{ab} expressed in terms of $\vec{E}, \vec{B}$, i.e. $F_{01} = -E_1/c$, $F_{12} = B_3$ etc. Let us look at the three cases in turn:

   i. Two indices equal ($a = b$, say): $\partial_a F_{ac} + \partial_a F_{ca} + \partial_c F_{aa} = 0$ is identically satisfied because $F_{aa} = 0$ and $F_{ac} + F_{ca} = 0$. Alternatively, this follows directly from the total anti-symmetry of (6).

   ii. All 3 indices spatial. Without loss of generality we can take $(a, b, c) = (1, 2, 3)$ (because any other choice is related to this one by anti-symmetry). Then we have

   $$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{21} = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \vec{\nabla}.\vec{B} = 0 \tag{7}$$

   iii. One index time, the others spatial, e.g. $(a, b, c) = (0, 1, 2)$:

   $$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = c^{-1}(\partial_t B_3 + \partial_1 E_2 - \partial_2 E_1) = c^{-1}(\vec{\nabla} \times \vec{E} + \partial_t \vec{B})_3 = 0 \tag{8}$$

   (and likewise for the other components).

2. The dual field strength tensor

The dual field strength tensor is defined by

$$\tilde{F}^{ab} = \tfrac{1}{2}\epsilon^{abcd}F_{cd} \quad , \tag{9}$$

with ϵ^{abcd} totally anti-symmetric, and $\epsilon^{0123} = -1$.

(a) As a consequence, one has

$$\partial_a \tilde{F}^{ab} = \tfrac{1}{2}\epsilon^{abcd}\partial_a F_{cd} = \tfrac{1}{2}\epsilon^{abcd}\partial_{[a}F_{cd]} \tag{10}$$

(because of the contraction with ϵ^{abcd} only the totally anti-symmetric part of $\partial_a F_{cd}$ contributes). Therefore

$$\partial_a \tilde{F}^{ab} = 0 \quad \Leftrightarrow \quad \partial_{[a}F_{cd]} = 0 \quad \Leftrightarrow \quad \partial_a F_{cd} + \text{cyclic permutations} = 0 \quad . \tag{11}$$

(b) Explicitly, the components of the dual field strength tensor are

$$\begin{aligned} \tilde{F}^{01} &= \tfrac{1}{2}\epsilon^{01cd}F_{cd} = \tfrac{1}{2}(\epsilon^{0123}F_{23} + \epsilon^{0132}F_{32}) = \epsilon^{0123}F_{23} = -F_{23} \\ \tilde{F}^{23} &= \tfrac{1}{2}\epsilon^{23cd}F_{cd} = \epsilon^{2301}F_{01} = \epsilon^{0123}F_{01} = -F_{01} \end{aligned} \tag{12}$$

etc. In terms of $\vec{E}$ and $\vec{B}$ this means

$$\tilde{F}^{01} = -B_1 \quad , \quad \tilde{F}^{23} = E_1/c \tag{13}$$

etc. The equation $\partial_a \tilde{F}^{ab} = 0$ can then be written as

$$\partial_a \tilde{F}^{a0} = \partial_i \tilde{F}^{i0} = \vec{\nabla}\cdot\vec{B} = 0 \tag{14}$$

$$\begin{aligned} \partial_a \tilde{F}^{aj} &= \partial_0 \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = -c^{-1}\partial_t B_j - c^{-1}\epsilon_{jik}\partial_i E_k \\ &= -\frac{1}{c}\left(\partial_t \vec{B} + \vec{\nabla}\times\vec{E}\right)_j = 0 \quad , \end{aligned} \tag{15}$$

which proves the assertion.

3. Lorentz Invariants

(a) Using $\epsilon^{ijl}\epsilon_{ijk} = 2\delta^l_k$, one has

$$I_1 = \tfrac{1}{4}F_{ab}F^{ab} = \tfrac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) = \tfrac{1}{2}\left(\vec{B}^2 - c^{-2}\vec{E}^2\right) \tag{16}$$

$$\begin{aligned} I_2 &= \tfrac{1}{4}F_{ab}\tilde{F}^{ab} = \tfrac{1}{4}\left(F_{0i}\tilde{F}^{0i} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij}\right) \\ &= \tfrac{1}{4}\left(2c^{-1}E_iB_i + \epsilon_{ijk}B_k c^{-1}\epsilon^{ijl}E_l\right) = c^{-1}\vec{E}\cdot\vec{B} \end{aligned} \tag{17}$$

(b) If $\vec{E} = 0$ in one inertial system, then $I_1 > 0$ and $I_2 = 0$ in all inertial systems, and thus $\vec{E}.\vec{B} = 0$ and $|\vec{E}/c| < |\vec{B}|$ in all inertial systems.

(c) When $F_{ab} = \partial_a A_b - \partial_b A_a$ one has

$$\partial_a \left(\epsilon^{abcd} A_b F_{cd} \right) = \epsilon^{abcd} (\partial_a A_b) F_{cd} + \epsilon^{abcd} A_b \partial_a F_{cd}$$
$$= \tfrac{1}{2} \epsilon^{abcd} (\partial_a A_b - \partial_b A_a) F_{cd} + \epsilon^{abcd} A_b \partial_{[a} F_{cd]} \tag{18}$$
$$= \tfrac{1}{2} \epsilon^{abcd} F_{ab} F_{cd} + 0 = F_{ab} \tilde{F}^{ab} \quad .$$

Therefore

$$C^a = \epsilon^{abcd} A_b F_{cd} \quad \Rightarrow \quad \partial_a C^a = F_{ab} \tilde{F}^{ab} \quad . \tag{19}$$

While $\partial_a C^a$ is gauge invariant, C^a itself is not. When $A_a \to A_a + \partial_a \Psi$, one has

$$C^a \to C^a + \epsilon^{abcd} (\partial_b \Psi) F_{cd} = C^a + \partial_b \left(\Psi \epsilon^{abcd} F_{cd} \right) \tag{20}$$

(where the last step again follows from the Bianchi identity).

(d) Using $\bar{F}_{ab} = \Lambda_a^{\ c} \Lambda_b^{\ d} F_{cd}$, one wants to compute the transformation of $\bar{F}_{ij}$ which contains the magnetic field components. To do this we need the matrix $\Lambda = (L^T)^{-1}$. Since L is symmetric, and for the inverse transformation one has $\alpha \to -\alpha$, Λ is the matrix

$$\left(\Lambda_a^{\ b} \right) = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{21}$$

With it one computes:

$$\begin{aligned} \bar{F}_{ij} &= \Lambda_i^{\ c} \Lambda_j^{\ d} F_{cd} = \Lambda_i^{\ 0} \Lambda_j^{\ l} F_{0l} + \Lambda_i^{\ k} \Lambda_j^{\ 0} F_{k0} + \Lambda_i^{\ k} \Lambda_j^{\ l} F_{kl} \\ &= \left(\Lambda_i^{\ 0} \Lambda_j^{\ l} - \Lambda_i^{\ l} \Lambda_j^{\ 0} \right) F_{0l} + \Lambda_i^{\ k} \Lambda_j^{\ l} F_{kl} \end{aligned} \tag{22}$$

such that

$$\bar{F}_{12} = \sinh \alpha F_{02} + \cosh \alpha F_{12} = -c^{-1} \gamma \beta E_2 + \gamma B_3 \tag{23}$$
$$\bar{F}_{23} = F_{23} \tag{24}$$
$$\bar{F}_{31} = -\sinh \alpha F_{03} + \cosh \alpha F_{31} = c^{-1} \gamma \beta E_3 + \gamma B_2 \tag{25}$$

from which one can read off the transformation of the magnetic field :

$$\begin{aligned} \bar{B}_1 &= B_1 \\ \bar{B}_2 &= \gamma B_2 + c^{-1} \beta \gamma E_3 \\ \bar{B}_3 &= \gamma B_3 - c^{-1} \beta \gamma E_2 \end{aligned} \tag{26}$$