## KFT Solutions 06

## 1. The Homogeneneous Maxwell-Equations

(a) For $G_{a b}=G_{[a b]}$ (totally) anti-symmetric, the definition

$$
\begin{equation*}
H_{a b c}=\partial_{a} G_{b c}+\partial_{b} G_{c a}+\partial_{c} G_{a b} \tag{1}
\end{equation*}
$$

implies that when one exchanges the two indices $a, b$, one gets

$$
\begin{equation*}
H_{b a c}=\partial_{b} G_{a c}+\partial_{a} G_{c b}+\partial_{c} G_{b a}=-\partial_{b} G_{c a}-\partial_{a} G_{b c}-\partial_{c} G_{a b}=-H_{a b c} \tag{2}
\end{equation*}
$$

(and likewise for the index pairs $a, c$ and $b, c$ ).
(b) One has

$$
\begin{equation*}
\partial_{a} F_{b c}=\partial_{a} \partial_{b} A_{c}-\partial_{a} \partial_{c} A_{b} \tag{3}
\end{equation*}
$$

etc. Because 2nd partial derivatives commute, the 1st term is symmetric in $a, b$, and the 2 nd term is symmetric in $a, c$. Therefore neither of them can appear in the totally anti-symmetric combination $\partial_{a} F_{b c}+\partial_{b} F_{c a}+\partial_{c} F_{a b}$. Thus this linear combination is identically zero,

$$
\begin{equation*}
F_{a b}=\partial_{a} A_{b}-\partial_{b} A_{a} \quad \Rightarrow \quad \partial_{a} F_{b c}+\partial_{b} F_{c a}+\partial_{c} F_{a b} \equiv 0 \quad(\text { Bianchi Identity }) \tag{4}
\end{equation*}
$$

One can of course also verify this explicitly,

$$
\begin{equation*}
\partial_{a} \partial_{b} A_{c}-\partial_{a} \partial_{c} A_{b}+\partial_{b} \partial_{c} A_{a}-\partial_{b} \partial_{a} A_{c}+\partial_{c} \partial_{a} A_{b}-\partial_{c} \partial_{b} A_{a}=0 \tag{5}
\end{equation*}
$$

(c) We consider the equations

$$
\begin{equation*}
\partial_{a} F_{b c}+\partial_{b} F_{c a}+\partial_{c} F_{a b}=0 \tag{6}
\end{equation*}
$$

with $F_{a b}$ expressed in terms of $\vec{E}, \vec{B}$, i.e. $F_{01}=-E_{1} / c, F_{12}=B_{3}$ etc. Let us look at the three cases in turn:
i. Two indices equal $(a=b$, say $): \partial_{a} F_{a c}+\partial_{a} F_{c a}+\partial_{c} F_{a a}=0$ is identically satisfied because $F_{a a}=0$ and $F_{a c}+F_{c a}=0$. Alternatively, this follows directly from the total anti-symmetry of (6).
ii. All 3 indices spatial. Without loss of gnerality we can take $(a, b, c)=$ $(1,2,3)$ (because any other choice is related to this one by anti-symmetry). Then we have

$$
\begin{equation*}
\partial_{1} F_{23}+\partial_{2} F_{31}+\partial_{3} F_{21}=\partial_{1} B_{1}+\partial_{2} B_{2}+\partial_{3} B_{3}=\vec{\nabla} \cdot \vec{B}=0 \tag{7}
\end{equation*}
$$

iii. One index time, the others spatial, e.g. $(a, b, c)=(0,1,2)$ :

$$
\begin{equation*}
\partial_{0} F_{12}+\partial_{1} F_{20}+\partial_{2} F_{01}=c^{-1}\left(\partial_{t} B_{3}+\partial_{1} E_{2}-\partial_{2} E_{1}\right)=c^{-1}\left(\vec{\nabla} \times \vec{E}+\partial_{t} \vec{B}\right)_{3}=0 \tag{8}
\end{equation*}
$$

(and likewise for the other components).

## 2. The dual field strength tensor

The dual field strength tensor is defined by

$$
\begin{equation*}
\tilde{F}^{a b}=\frac{1}{2} \epsilon^{a b c d} F_{c d} \tag{9}
\end{equation*}
$$

with $\epsilon^{a b c d}$ totally anti-symmetric, and $\epsilon^{0123}=-1$.
(a) As a consequence, one has

$$
\begin{equation*}
\partial_{a} \tilde{F}^{a b}=\frac{1}{2} \epsilon^{a b c d} \partial_{a} F_{c d}=\frac{1}{2} \epsilon^{a b c d} \partial_{[a} F_{c d]} \tag{10}
\end{equation*}
$$

(because of the contraction with $\epsilon^{a b c d}$ only the totally anti-symmetric part of $\partial_{a} F_{c d}$ contributes). Therefore

$$
\begin{equation*}
\partial_{a} \tilde{F}^{a b}=0 \quad \Leftrightarrow \quad \partial_{[a} F_{c d]}=0 \quad \Leftrightarrow \quad \partial_{a} F_{c d}+\text { cyclic permutations }=0 . \tag{11}
\end{equation*}
$$

(b) Explicitly, the components of the dual field strength tensor are

$$
\begin{align*}
& \tilde{F}^{01}=\frac{1}{2} \epsilon^{01 c d} F_{c d}=\frac{1}{2}\left(\epsilon^{0123} F_{23}+\epsilon^{0132} F_{32}\right)=\epsilon^{0123} F_{23}=-F_{23} \\
& \tilde{F}^{23}=\frac{1}{2} \epsilon^{23 c d} F_{c d}=\epsilon^{2301} F_{01}=\epsilon^{0123} F_{01}=-F_{01} \tag{12}
\end{align*}
$$

etc. In terms of $\vec{E}$ and $\vec{B}$ this means

$$
\begin{equation*}
\tilde{F}^{01}=-B_{1} \quad, \quad \tilde{F}^{23}=E_{1} / c \tag{13}
\end{equation*}
$$

etc. The equation $\partial_{a} \tilde{F}^{a b}=0$ can then be written as

$$
\begin{align*}
\partial_{a} \tilde{F}^{a 0} & =\partial_{i} \tilde{F}^{i 0}=\vec{\nabla} \cdot \vec{B}=0  \tag{14}\\
\partial_{a} \tilde{F}^{a j} & =\partial_{0} \tilde{F}^{0 j}+\partial_{i} \tilde{F}^{i j}=-c^{-1} \partial_{t} B_{j}-c^{-1} \epsilon_{j i k} \partial_{i} E_{k} \\
& =-\frac{1}{c}\left(\partial_{t} \vec{B}+\vec{\nabla} \times \vec{E}\right)_{j}=0, \tag{15}
\end{align*}
$$

which proves the assertion.

## 3. Lorentz Invariants

(a) Using $\epsilon^{i j l} \epsilon_{i j k}=2 \delta_{k}^{l}$, one has

$$
\begin{align*}
I_{1} & =\frac{1}{4} F_{a b} F^{a b}=\frac{1}{4}\left(F_{0 i} F^{0 i}+F_{i 0} F^{i 0}+F_{i j} F^{i j}\right)=\frac{1}{2}\left(\vec{B}^{2}-c^{-2} \vec{E}^{2}\right)  \tag{16}\\
I_{2} & =\frac{1}{4} F_{a b} \tilde{F}^{a b}=\frac{1}{4}\left(F_{0 i} \tilde{F}^{0 i}+F_{i 0} \tilde{F}^{i 0}+F_{i j} \tilde{F}^{i j}\right) \\
& =\frac{1}{4}\left(2 c^{-1} E_{i} B_{i}+\epsilon_{i j k} B_{k} c^{-1} \epsilon^{i j l} E_{l}\right)=c^{-1} \vec{E} \cdot \vec{B} \tag{17}
\end{align*}
$$

(b) If $\vec{E}=0$ in one inertial system, then $I_{1}>0$ and $I_{2}=0$ in all inertial systems, and thus $\vec{E} \cdot \vec{B}=0$ and $|\vec{E} / c|<|\vec{B}|$ in all inertial systems.
(c) When $F_{a b}=\partial_{a} A_{b}-\partial_{b} A_{a}$ one has

$$
\begin{align*}
\partial_{a}\left(\epsilon^{a b c d} A_{b} F_{c d}\right) & =\epsilon^{a b c d}\left(\partial_{a} A_{b}\right) F_{c d}+\epsilon^{a b c d} A_{b} \partial_{a} F_{c d} \\
& =\frac{1}{2} \epsilon^{a b c d}\left(\partial_{a} A_{b}-\partial_{b} A_{a}\right) F_{c d}+\epsilon^{a b c d} A_{b} \partial_{[a} F_{c d]}  \tag{18}\\
& =\frac{1}{2} \epsilon^{a b c d} F_{a b} F_{c d}+0=F_{a b} \tilde{F}^{a b} .
\end{align*}
$$

Therefore

$$
\begin{equation*}
C^{a}=\epsilon^{a b c d} A_{b} F_{c d} \quad \Rightarrow \quad \partial_{a} C^{a}=F_{a b} \tilde{F}^{a b} \tag{19}
\end{equation*}
$$

While $\partial_{a} C^{a}$ is gauge invariant, $C^{a}$ itself is not. When $A_{a} \rightarrow A_{a}+\partial_{a} \Psi$, one has

$$
\begin{equation*}
C^{a} \rightarrow C^{a}+\epsilon^{a b c d}\left(\partial_{b} \Psi\right) F_{c d}=C^{a}+\partial_{b}\left(\Psi \epsilon^{a b c d} F_{c d}\right) \tag{20}
\end{equation*}
$$

(where the last step again follows from the Bianchi identity).
(d) Using $\bar{F}_{a b}=\Lambda_{a}^{c} \Lambda_{b}^{d} F_{c d}$, one wants to compute the transformation of $\bar{F}_{i j}$ which contains the magnetic field components. To do this we need the matrix $\Lambda=\left(L^{T}\right)^{-1}$. Since $L$ is symmetric, and for the inverse transformation one has $\alpha \rightarrow-\alpha, \Lambda$ is the matrix

$$
\left(\Lambda_{a}^{b}\right)=\left(\begin{array}{cccc}
\cosh \alpha & \sinh \alpha & 0 & 0  \tag{21}\\
\sinh \alpha & \cosh \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

With it one computes:

$$
\begin{align*}
\bar{F}_{i j} & =\Lambda_{i}{ }^{c} \Lambda_{j}^{d} F_{c d}=\Lambda_{i}{ }^{0} \Lambda_{j}^{l} F_{0 l}+\Lambda_{i}^{k} \Lambda_{j}^{0} F_{k 0}+\Lambda_{i}^{k} \Lambda_{j}^{l} F_{k l} \\
& =\left(\Lambda_{i}^{0} \Lambda_{j}^{l}-\Lambda_{i}^{l} \Lambda_{j}^{0}\right) F_{0 l}+\Lambda_{i}^{k} \Lambda_{j}^{l} F_{k l} \tag{22}
\end{align*}
$$

such that

$$
\begin{align*}
\bar{F}_{12} & =\sinh \alpha F_{02}+\cosh \alpha F_{12}=-c^{-1} \gamma \beta E_{2}+\gamma B_{3}  \tag{23}\\
\bar{F}_{23} & =F_{23}  \tag{24}\\
\bar{F}_{31} & =-\sinh \alpha F_{03}+\cosh \alpha F_{31}=c^{-1} \gamma \beta E_{3}+\gamma B_{2} \tag{25}
\end{align*}
$$

from which one can read off the transformation of the magnetic field :

$$
\begin{align*}
& \bar{B}_{1}=B_{1} \\
& \bar{B}_{2}=\gamma B_{2}+c^{-1} \beta \gamma E_{3} \\
& \bar{B}_{3}=\gamma B_{3}-c^{-1} \beta \gamma E_{2} \tag{26}
\end{align*}
$$

