# Solutions to Assignments 08

1. Complex Scalar Field II: Phase Invariance and Noether-Theorem

   (a) If the potential is a function of $\Phi^*\Phi$, both the potential and the derivative terms of the Lagrangian are obviously invariant under

$$\Phi(x) \to \mathrm{e}^{i\theta}\Phi(x) \quad , \quad \Phi^*(x) \to \mathrm{e}^{-i\theta}\Phi^*(x) \tag{1}$$

   for constant $\theta$, since in this case the derivatives transform the same way, i.e.

$$\partial_a\Phi(x) \to \mathrm{e}^{i\theta}\partial_a\Phi(x) \quad , \quad \partial_a\Phi^*(x) \to \mathrm{e}^{-i\theta}\partial_a\Phi^*(x) \tag{2}$$

   (b) Infinitesimally, one has

$$\Delta\Phi = i\theta\Phi \quad , \quad \Delta\Phi^* = -i\theta\Phi^* \quad , \tag{3}$$

   and therefore the corresponding Noether current is

$$J^a_\Delta = \frac{\partial L}{\partial(\partial_a\Phi)}\Delta\Phi + \frac{\partial L}{\partial(\partial_a\Phi^*)}\Delta\Phi^* = -(i\theta/2)(\Phi\partial^a\Phi^* - \Phi^*\partial^a\Phi) \tag{4}$$

   where (as usual) $\partial^a = \eta^{ab}\partial_b$. Calculating its divergence, one finds (ignoring the irrelevant constant prefactor, and using the equations of motion)

$$\begin{aligned}\partial_a(\Phi\partial^a\Phi^* - \Phi^*\partial^a\Phi) &= \partial_a\Phi\partial^a\Phi^* + \Phi\Box\Phi^* - \partial_a\Phi^*\partial^a\Phi - \Phi^*\Box\Phi \\ &= \Phi\Box\Phi^* - \Phi^*\Box\Phi = 2(\Phi\partial W/\partial\Phi - \Phi^*\partial W/\partial\Phi^*)\end{aligned} \tag{5}$$

   This is not (and should not be) zero in general, but it is zero precisely when $W = W(\Phi^*\Phi)$. Indeed, in that case one has

$$\partial W(\Phi^*\Phi)/\partial\Phi = W'(\Phi^*\Phi)\Phi^* \quad , \quad \partial W(\Phi^*\Phi)/\partial\Phi^* = W'(\Phi^*\Phi)\Phi \quad , \tag{6}$$

   and therefore

$$\Phi\partial W/\partial\Phi - \Phi^*\partial W/\partial\Phi^* = W'(\Phi^*\Phi)\,(\Phi\Phi^* - \Phi^*\Phi) = 0 \quad . \tag{7}$$

2. Complex Scalar Field III: Gauge Invariance and Minimal Coupling

   (a) Under

$$\Phi(x) \to \mathrm{e}^{i\theta(x)}\Phi(x) \quad , \quad \Phi^*(x) \to \mathrm{e}^{-i\theta(x)}\Phi^*(x) \quad , \quad A_a(x) \to A_a(x) + \partial_a\theta(x) \tag{8}$$

   the partial derivative transforms as

$$\partial_a\Phi \to \partial_a(\mathrm{e}^{i\theta}\Phi) = \mathrm{e}^{i\theta}(\partial_a\Phi + i(\partial_a\theta)\Phi) \tag{9}$$

Therefore the covariant derivative

$$D_a\Phi = \partial_a\Phi - iA_a\Phi \quad , \quad D_a\Phi^* = \partial_a\Phi^* + iA_a\Phi^* \ . \tag{10}$$

transforms as

$$\begin{aligned} D_a\Phi &\to \mathrm{e}^{i\theta}(\partial_a\Phi + i(\partial_a\theta)\Phi) - i\mathrm{e}^{i\theta}A_a\Phi - i\mathrm{e}^{i\theta}(\partial_a\theta)\Phi \\ &= \mathrm{e}^{i\theta}(\partial_a\Phi - iA_a\Phi) = \mathrm{e}^{i\theta}D_a\Phi \end{aligned} \tag{11}$$

Likewise

$$D_a\Phi^* \to \mathrm{e}^{-i\theta}D_a\Phi^* \ . \tag{12}$$

(b) It is now obvious that the action

$$S[\Phi, A] = \int d^4x \left(-\tfrac{1}{2}\eta^{ab}D_a\Phi D_b\Phi^* - W(\Phi\Phi^*)\right) \tag{13}$$

is gauge invariant.

(c) The action is

$$S = S_{\mathrm{Maxwell}}[A] + S[\Phi, A] = \int d^4x(-\tfrac{1}{4}F^2) + S[\Phi, A] \ . \tag{14}$$

The equations of motion for $\Phi$ and $\Phi^*$ are simply the covariant versions of the equations of motion from Exercise 07.2, namely

$$D^aD_a\Phi = 2\partial W/\partial\Phi^* \quad , \quad D^aD_a\Phi^* = 2\partial W/\partial\Phi \ . \tag{15}$$

Variation with respect to $A$ leads to

$$\delta S = \int d^4x \left(\partial_a F^{ab} + J^b\right)\delta A_b \tag{16}$$

where $J^b$ arises from the variation of the gauge field in the covariant derivatives in $S[\phi, A]$. From the variation of $A_a$ in $D_a\Phi$ one finds a term

$$-(1/2)\eta^{ab}(-i)(\delta A_a)\Phi D_b\Phi^* = (i/2)\Phi D^b\Phi^*\delta A_b \ . \tag{17}$$

Combining this with the analogous term that one obtains from the variation of $A_a$ in $D_b\Phi^*$, one finds

$$J^b = (i/2)\left(\Phi D^b\Phi^* - \Phi^* D^b\Phi\right) \tag{18}$$

The equations of motion $\partial_a F^{ab} + J^b = 0$ imply (and therefore require) that $\partial_b J^b = 0$. Let us show that this equation is satisfied as a consequence of the equations of motion for $\Phi$.

First of all, we have

$$\partial_b(\Phi D^b\Phi^*) = \partial_b\Phi D^b\Phi^* + \Phi\partial_b D^b\Phi^* \ . \tag{19}$$

Adding and subtracting $+iA_b\Phi$, we can write this as

$$\partial_b(\Phi D^b\Phi^*) = D_b\Phi D^b\Phi^* + \Phi D_b D^b\Phi^* \ . \tag{20}$$

Since the first term is invariant under the exchange $\Phi \leftrightarrow \Phi^*$, one finds

$$\partial_b\left(\Phi D^b\Phi^* - \Phi^* D^b\Phi\right) = \Phi D_b D^b\Phi^* - \Phi^* D_b D^b\Phi \tag{21}$$

Note that this is just the covariant version of the divergence of the Noether current in Exercise 1, and the remaining step in the proof that this vanishes for a solution to the equations of motion is now identical to that in Exercise 1.