Solutions 09

1. Noether Energy-Momentum Tensor

   The claim follows from

$$
\begin{aligned}
\frac{d}{dx^a}\Theta^a_{\ b} &= \frac{d}{dx^a}\left(-\frac{\partial L}{\partial(\partial_a\Phi^A)}\partial_b\Phi^A + \delta^a_b L\right) \\
&= -\left(\frac{d}{dx^a}\frac{\partial L}{\partial(\partial_a\Phi^A)}\right)\partial_b\Phi^A - \frac{\partial L}{\partial(\partial_a\Phi^A)}\partial_a\partial_b\Phi^A + \frac{d}{dx^b}L \\
&= \left(\frac{\partial L}{\partial\Phi^A} - \frac{d}{dx^a}\frac{\partial L}{\partial(\partial_a\Phi^A)}\right)\partial_b\Phi^A + \frac{\partial}{\partial x^b}L
\end{aligned}
\tag{1}
$$

2. Noether Energy-Momentum Tensor for a Scalar Field

   The action is

$$
\begin{aligned}
S[\phi] &= \int d^4x\left(-\tfrac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - V(\phi)\right) \equiv \int d^4x\left(-\tfrac{1}{2}(\partial\phi)^2 - V(\phi)\right) \\
\Rightarrow\quad \delta S[\phi] &= \int d^4x\left(-\eta^{ab}\partial_a\phi\partial_b\delta\phi - V'(\phi)\delta\phi\right) \\
&= \int d^4x\left(\eta^{ab}\partial_a\partial_b\phi - V'(\phi)\right)\delta\phi = \int d^4x\left(\Box\phi - V'(\phi)\right)\delta\phi \\
\Rightarrow\quad \Box\phi &= V'(\phi)
\end{aligned}
\tag{2}
$$

   The energy-momentum tensor is

$$
\Theta_{ab} = -\frac{\partial L}{\partial(\partial^a\phi)}\partial_b\phi + \eta_{ab}L = \partial_a\phi\partial_b\phi - \eta_{ab}(\tfrac{1}{2}(\partial\phi)^2 + V(\phi)) \ . \tag{3}
$$

   The claim follows from

$$
\begin{aligned}
\partial^a\Theta_{ab} &= (\Box\phi)\partial_b\phi + \partial_a\phi\partial^a\partial_b\phi - \partial_b(\tfrac{1}{2}(\partial\phi)^2 + V(\phi)) \\
&= (\Box\phi)\partial_b\phi + \partial_a\phi\partial^a\partial_b\phi - \partial_b\partial^a\phi\partial_a\phi - V'(\phi)\partial_b\phi \\
&= (\Box\phi - V'(\phi))\partial_b\phi \ .
\end{aligned}
\tag{4}
$$

3. Maxwell Energy-Momentum Tensor

   (a) The canonical (Noether) energy-momentum tensor is

$$
\Theta_{ab} = -\frac{\partial L}{\partial(\partial^a A_c)}\partial_b A_c + \eta_{ab}L = F_a^{\ c}\partial_b A_c - \tfrac{1}{4}\eta_{ab}F_{cd}F^{cd} \ , \tag{5}
$$

   the lack of gauge invariance arising from the translational variation $\delta_T A_a = -\epsilon^b\partial_b A_c$. This situation can be improved by first of all manipulating Θ_{ab} as

$$
\Theta_{ab} = F_a^{\ c}F_{bc} - \tfrac{1}{4}\eta_{ab}F_{cd}F^{cd} + F_{ac}\partial^c A_b \equiv T_{ab} + F_{ac}\partial^c A_b \ . \tag{6}
$$

   Here the first two terms are already nice and gauge invariant. The last term can be written as a sum of two terms,

$$
F_{ac}\partial^c A_b = \partial^c(F_{ac}A_b) - (\partial^c F_{ac})A_b \ . \tag{7}
$$

The first of these is identically conserved because of $F_{ac} = -F_{ca}$.

$$\partial^a \partial^c (F_{ac} A_b) = 0 \quad \text{identically} \quad . \tag{8}$$

The second term in (7) is identically zero on solutions, $(\partial^c F_{ac}) A_b = 0$. Removing both these terms, one can define a new (and vastly improved) energy-momentum tensor T_{ab} by

$$T_{ab} = F_{ac} F_b{}^c - \tfrac{1}{4} \eta_{ab} F_{cd} F^{cd} \quad . \tag{9}$$

(b)

$$\Delta_T A_c = \delta_T A_c + \partial_c (\epsilon^b A_b) = -\epsilon^b F_{bc} \quad , \tag{10}$$

is a variation, and therefore

$$\Delta_T F_{cd} = \partial_c \Delta_T A_d - \partial_d \Delta_T A_c = -\epsilon^b (\partial_c F_{bd} - \partial_d F_{bc}) \quad . \tag{11}$$

The Bianchi identity implies

$$\partial_c F_{bd} - \partial_d F_{bc} = -\partial_c F_{db} - \partial_d F_{bc} = +\partial_b F_{cd} \tag{12}$$

and therefore

$$\Delta_T F_{cd} = -\epsilon^b \partial_b F_{cd} = \delta_T F_{cd} \quad . \tag{13}$$

(c) One has

$$\Delta_T L = \delta_T L = \frac{d}{dx^a} (-\epsilon^a L) \quad . \tag{14}$$

Therefore the conserved currents are

$$J^a_{\Delta_T} = \frac{\partial L}{\partial(\partial_a A_c)} \Delta_T A_c + \epsilon^a L = \epsilon^b (F^{ac} F_{bc} - \tfrac{1}{4} \delta^a_b F_{cd} F^{cd}) = T^a_b \epsilon^b \quad . \tag{15}$$

(d) T_{ab} is symmetric:

$$F_{bc} F_a{}^c = F_a{}^c F_{bc} = F_{ac} F_b{}^c \tag{16}$$

(MM^t is symmetric for any matrix M ...). Therefore $T_{ab} = T_{ba}$.

T_{ab} is traceless:

In D spacetime dimensions one has (the expression for T_{ab} is valid for any D)

$$T^a_a = \eta^{ab} T_{ab} = \eta^{ab} F_{ac} F_b{}^c - \tfrac{1}{4} \eta^{ab} \eta_{ab} F_{cd} F^{cd} = F_{ac} F^{ac} - \frac{D}{4} F_{cd} F^{cd} \quad , \tag{17}$$

so this is zero precisely for $D = 4$.

(e) From (9) we find

$$\partial^a T_{ab} = (\partial^a F_{ac}) F_b{}^c + F_{ac} \partial^a F_b{}^c - \tfrac{1}{2} (\partial_b F_{cd}) F^{cd} \ . \tag{18}$$

The Maxwell equations imply that the first term on the right-hand side is zero. In order to be able to combine the remaining terms, we relabel and raise/lower the indices such that

$$F_{ac} \partial^a F_b{}^c - \tfrac{1}{2} (\partial_b F_{cd}) F^{cd} = F^{ac} \partial_a F_{bc} - \tfrac{1}{2} (\partial_b F_{ac}) F^{ac} = F^{ac} (\partial_a F_{bc} - \tfrac{1}{2} \partial_b F_{ac}) \ . \tag{19}$$

Since $F^{ac} = -F^{ca}$, only the anti-symmetric part of $\partial_a F_{bc}$ contributes, and therefore we anti-symmetrise explicitly, to find

$$F^{ac} (\partial_a F_{bc} - \tfrac{1}{2} \partial_b F_{ac}) = \tfrac{1}{2} F^{ac} (\partial_a F_{bc} - \partial_c F_{ba} - \partial_b F_{ac}) \tag{20}$$

Finally, by the homogeneous Maxwell equations, the term in brackets is zero,

$$\partial_a F_{bc} - \partial_c F_{ba} - \partial_b F_{ac} = \partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0 \ . \tag{21}$$