Solutions to Assignments 01b

1. Geometry : Analytic Minkowski-Geometry

   Because of Lorentz invariance one can choose to work in the rest frame in which the timelike vector v has the components $v = (v^0, 0, 0, 0)$, with $v^0 \neq 0$.

   (a) Show that $w.v = 0$ implies that w is spacelike, $w^2 > 0$:

   In this case one has

   $$v.w = \eta_{ab} v^a w^b = -v^0 w^0 = 0 \quad \Rightarrow \quad w^0 = 0 \quad \Rightarrow \quad w = (0, w^1, w^2, w^3) \quad (1)$$

   and thus w is manifestly spacelike,

   $$w^2 = \eta_{ab} w^a w^b = (w^1)^2 + (w^2)^2 + (w^3)^2 > 0 \ . \quad (2)$$

   (b) Show that any timelike v can be written as a sum of two lightlike vectors:

   Decompose v as

   $$v = (v^0, 0, 0, 0) = \tfrac{1}{2}(v^0, v^0, 0, 0) + \tfrac{1}{2}(v^0, -v^0, 0, 0) \quad (3)$$

   Both vectors on the right-hand side are lightlike. Indeed, for any vector $w = (w^0, w^1 = \pm w^0, 0, 0)$ one has

   $$\eta_{ab} w^a w^b = -(w^0)^2 + (w^1)^2 = -(w^0)^2 + (w^0)^2 = 0 \ . \quad (4)$$

   (c) Sum of spacelike vectors and sum of timelike vectors

   The sum of two spacelike vectors is *not* necessarily spacelike. As a counterexample, consider the vectors $u = (a, b, 0, 0)$ and $v = (a, -b, 0, 0)$; these are spacelike for $|b| > |a|$ but their sum $u + v = (2a, 0, 0, 0)$ is clearly timelike. This shows that one should not think that for spacelike vectors Minkowski geometry reduces to Euclidean geometry.

   The sum of two timelike vectors is *not* necessarily timelike. As a counterexample, consider the vectors $u = (a, b, 0, 0)$ and $v = (-a, b, 0, 0)$; these are timelike for $|b| < |a|$ but their sum $u + v = (0, 2b, 0, 0)$ is clearly spacelike. What *is* true is that the sum of two future-pointing ($u^0 > 0, v^0 > 0$) timelike vectors is future-pointing and timelike.