Supergravity Solitons

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The purpose of these lectures is to find and study a class of interesting classical solutions of supergravity theories, namely solitons of type IIA/B supergravity carrying Ramond-Ramond charges (so-called *RR p-branes*) and solitons of eleven-dimensional supergravity (*M-branes*).

The motivation for doing this is to gain some insight into the non-perturbative properties of string theory and particularly to study in string theory the physics of black holes. We will barely be able to scratch the surface of this subject, but some aspects will be covered in the other lectures of this School. How to actually incorporate some of these supergravity solitons into string theory, in terms of D-branes, is the subject of the parallel lectures by Marco Serone. Various applications of D-branes and p-branes to string dualities will appear in the lectures by K.S. Narain and Massimo Bianchi.

Over the years, a large number of individuals and groups have contributed to the understanding of supergravity solitons. Rather than referring you to the (copious) original literature I have listed some review articles at the end which I have found particularly helpful in preparing these lectures.

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An updated and corrected version of these notes will be available at
`http://www.ictp.trieste.it/~mblau/`

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[1] This section has been contributed by Martin O'Loughlin (SISSA)

1 Introduction and Motivation

The primary purpose of the first lecture is to understand

- why one could or should be interested in certain classical solutions of supergravity,
- what kinds of 'interesting' solutions one might hope or expect to find, and
- how one is actually going to go about finding these solutions,

and we will deal with these topics in that order.

The supergravity action and equations of motion are quite complicated, involving a large number of non-linearly and non-minimally coupled fields. Thus, a brute force approach to solving these equations is not a particularly promising strategy.

Indeed, already in standard four-dimensional general relativity this is not feasible and before trying to solve the Einstein equations one makes numerous simplifying assumptions. These are, typically, physically motivated assumptions about the symmetries of the solutions.

Here, we will proceed in the same way. But, before being able to do this, we need to understand why we are interested in certain classical supergravity configurations and what sort of solutions to expect. Once we have done that, we will be able to reduce the classical supergravity equations of motion to a tractable and solvable system of equations with, nevertheless, a rich and interesting spectrum of solutions.

1.1 A simple explanatory example

To get us started let us consider the prototypical example of the Reissner-Nordstrøm (RN) black hole as a solution to Einstein gravity coupled to a $U(1)$ gauge field the action for which has the form,

$$S = \frac{1}{G_N} \int d^4x \sqrt{-g}(R - \frac{1}{4}F^2) \tag{1.1}$$

We consider solutions with a fixed charge parameter Q and mass M. The metric is,

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2 \tag{1.2}$$

with a purely electric field pointing in the radial direction as one has for a point source of charge Q in flat space electromagnetism.

This solution is a non-perturbative configuration as it cannot be found by solving the Einstein equations linearised around flat space. It is a non-trivial solution to the full non-linear equations. The solution is not stable under Hawking evaporation for $M > Q$ as it has a finite positive temperature. However for the special case for which the mass equals the charge, the temperature is zero and the solution becomes stable (soliton-like). This is the typical behaviour of a black hole with charge and the stable extremal configuration is typical of BPS solitons as we will see in the following. Indeed the extremal ($Q = M$) RN space-time preserves supersymmetry. As we will indeed see all BPS solitons have an associated charge and it is the equality of the mass and this charge that in general guarantees preserved supersymmetry and the consequent stability. The extremal RN metric is simply

$$ds^2 = -(1 - \frac{Q}{r})^2 dt^2 + (1 - \frac{Q}{r})^{-2} dr^2 + r^2 d\Omega^2. \tag{1.3}$$

Changing the radial coordinate to $R = r - Q$ the metric is,

$$ds^2 = -(1 + \frac{Q}{R})^{-2} dt^2 + (1 + \frac{Q}{R})^2 (dR^2 + R^2 d\Omega^2). \tag{1.4}$$

In particular we see that the horizon originally at $r = Q$ is now at $R = 0$ which will be a general feature of the coordinate system we use for the supergravity solitons (p-branes).

We will see that the form of the metrics for all p-branes are simple generalisations of the metric for the RN black hole. The charge and masses appear in a similar way, and there are usually two types of factors, those that multiply the world-volume and those that rescale the space transverse to the world-volume. Moreover, just as here, these factors are expressed in terms of harmonic functions on the transverse space.

We will explain in detail the general ansatz after discussing the simplifying role played by supersymmetry in the search for solutions. In the simple case discussed here the factor rescaling the world-volume rescales only the time co-ordinate. The black hole soliton has a one dimensional world-line and is a particle (or 0-brane in modern parlance) with no spatial extent.

The fact that black holes can be seen as solitonic solutions in string theory and the related fact that BPS solitons can be equivalently described in perturbative string theory using D-brane techniques leads us to the observation that string theory can be used to obtain non-trivial facts about black hole physics. This was one of the first applications of D-brane techniques and was also later the source of the by now much studied AdS/CFT correspondence.

1.2 Why (Solitons and Supersymmetry)

Why Solitons?

So far in this school, we have studied perturbative string theory. Now one time-honoured way to try to get a handle on certain non-perturbative aspects of a field theory is via the study of solitons.

Roughly speaking, solitons are static, localized, finite energy solutions of the classical equations of motion of a non-linear field theory. Famous examples are the kink solution of sine-Gordon theory, the solitary wave solutions of the KdV equation (from which the name 'soliton' derives), or monopoles in Yang-Mills-Higgs systems. If you have not encountered such objects before, don't worry. Most of the things we will need we will derive ourselves. If you already have, the better.

If the solitons are localized at or around a point in space, they have many of the characteristics of a point particle. More generally, however, one can also find extended solitons like strings (vortices) or membranes (domain walls), and it is indeed such configurations that we will typically find in supergravity theories.

Solitons are usually characterized by the following properties:

1. They probe the non-linear structure of a theory because they are solutions of the non-linear field equations which cannot be found by perturbing solutions of the linearized field equations. In this sense, solitons are non-perturbative.

2. Typically, stability of these solutions is ensured by the fact that they carry a non-zero (topological) conserved quantum number or charge which would be zero in the perturbative sector.

3. They probe the non-perturbative structure of the theory because their mass or mass density is usually found to be inversely proportional to some positive power of a dimensionless coupling constant. Some such behaviour is required because otherwise one could take the weak coupling limit and hence find these solutions in perturbation theory - contradicting property 1.

   Hence solitons become very massive at weak coupling and the quantum effects due to exchanges of solitons will be non-perturbative effects vanishing to all orders in perturbation theory.

While this is all very nice and good, it does not really tell us yet to what use the solitons can be put and how to get a handle on their quantum properties. However, continuing this line of thought let us make the following observations:

4. If the classical mass formula can be trusted in the quantum theory, then at strong coupling the solitons become very light. In fact, at sufficiently strong coupling they are likely to be the lightest objects in the theory and they should therefore dominate the strong coupling low energy dynamics. As a consequence they are ideally suited for studying the strong coupling and non-perturbative aspects of a theory.

Why Supersymmetry? An Intermezzo on Central Charges and BPS states

However, this raises the question when, if ever, the classical mass formula can be trusted. Here supersymmetry enters the stage for the first time and comes to the rescue. Extended supersymmetry algebras with central charges have special massive representations, so-called short multiplets. The states in these representations, the BPS states, are annihilated (left invariant) by some of the generators of the supersymmetry algebra. They are characterized by the fact that they saturate the Bogomolny'i bound, an inequality between its mass and its (topological central) charge.

To see how this comes about, let us for example consider the centrally extended $N = 2$ supersymmetry algebra in four dimensions (the situation for higher N and other dimensions is similar). In the usual dotted-undotted notation, a Majorana spinor Q can be written in terms of Weyl spinors $Q_\alpha, Q_{\dot\alpha}$ as

$$Q = \left(\begin{array}{c} Q_\alpha \\ Q^{\dot\alpha} \end{array} \right) \quad , \tag{1.5}$$

where $Q^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}Q_{\dot\beta}$, $Q_{\dot\beta} = Q^*_\beta$ and $\epsilon^{\dot\alpha\dot\beta} = i\sigma_2$. Then the extended supersymmetry algebra, which has a single central charge Z, takes the form

$$\begin{aligned} \{Q^I_\alpha, Q_{\dot\beta J}\} &= \sigma^\mu_{\alpha\dot\beta}P_\mu\delta^I{}_J \\ \{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta}\epsilon^{IJ}Z \\ \{Q_{\dot\alpha I}, Q_{\dot\beta J}\} &= \epsilon_{\dot\alpha\dot\beta}\epsilon_{IJ}Z \quad . \end{aligned} \tag{1.6}$$

Now let us go to a new basis

$$a^\pm_\alpha = \tfrac{1}{\sqrt{2}}(Q^1_\alpha \pm \epsilon_{\alpha\beta}Q^{2*}_\beta) \quad . \tag{1.7}$$

In this basis the algebra takes the form of a fermionic oscillator algebra. In the rest frame $P_\mu = (M,0,0,0)$ one has

$$\begin{aligned} \{a^+_\alpha, a^{+\dagger}_\beta\} &= \delta_{\alpha\beta}(M+Z) \\ \{a^-_\alpha, a^{-\dagger}_\beta\} &= \delta_{\alpha\beta}(M-Z) \quad , \end{aligned} \tag{1.8}$$

with all other anticommutators vanishing,

$$\{a_\alpha^+, a_\beta^+\} = \{a_\alpha^-, a_\beta^-\} = \{a_\alpha^+, a_\beta^-\} = 0 \ . \tag{1.9}$$

From these relations we can immediately read off two things. The first is that unitarity imposes the requirement (Bogomolny'i bound)

$$M \geq |Z| \ . \tag{1.10}$$

Moreover, when $M = |Z|$, one of the set of oscillators is realized trivially. This means that these operators act trivially on all the states of the representation. In other words, these states are invariant under half of the supersymmetry algebra, they are half-BPS states.

At this point the algebra resembles that of a massless representation and hence the dimension of the representation is greatly reduced. In the present case, a generic massive representation would have dimension $2^{2N} = 16$ (a long multiplet), whereas the dimension of a massless representation and of the special massive BPS representations is $2^N = 4$ (short multiplet).

Even though both mass and charge may undergo renormalization, this definite mass-charge relationship is expected to be protected from quantum corrections. Indeed, if it were violated, then at a certain coupling 12 new states would have to appear out of nowhere and quantum corrections are not expected to produce these new degrees of freedom.

Solitons and Supersymmetry

Now let us go back to the problem at hand. We have just argued that solitonic BPS states may provide a viable starting point for exploring the behaviour of a theory at strong coupling.

Rephrased in terms of classical solutions of the equations of motion whose quantum theory we would like to understand, this means that we should look for field configurations solving not only the classical equations of motion but also being invariant under some of the supersymmetry variations of these fields. Typically, the latter requirement is quite restrictive and satisfying it brings one a long way towards solving the equations of motion themselves.

The upshot of this is a rather drastic simplification of the original problem as typically the supersymmetry charges are first order differential operators and thus the BPS condition (of supersymmetry invariance) is a first order differential equation - to be contrasted with the second order differential equations of motion. We will see examples of this (from supergravity) below.

By the way, this should remind you of the relation between the second order Yang-Mills equations and the first order instanton equations. Indeed, the latter can be derived from the BPS condition in the Euclidean version of $N=2$ supersymmetric Yang-Mills theories where

$$\delta\Psi \sim F_{\mu\nu}\Gamma^{\mu\nu}\epsilon + \ldots \quad . \tag{1.11}$$

Here Ψ is a Dirac spinor, ϵ is the supersymmetry parameter, and the dots refer to terms involving the scalar fields of the theory.

Clearly there is no non-trivial gauge field configuration satisfying $\delta\Psi = 0$ for all ϵ. However, if we impose the chirality condition $\gamma^5\epsilon = \pm\epsilon$, then we have

$$\delta\Psi = \tfrac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})\Gamma^{\mu\nu}\epsilon \tag{1.12}$$

where

$$\tilde{F}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho} \quad . \tag{1.13}$$

Thus $\delta\Psi = 0$ is solved by the (anti-)instanton configurations

$$F_{\mu\nu} \pm \tilde{F}_{\mu\nu} = 0 \quad , \tag{1.14}$$

which, as anticipated, is a first-order differential equation. Such configurations, supplemented by $\Psi = 0$ (and setting the scalar field to zero as well) preserve (i.e. are left invariant by) half of the original supersymmetry parameters and break the other half.

Of course these are instantons (finite action solutions of the Euclidean equations of motion) and not solitons (static finite energy solutions), but the principle is the same. In fact, one could view these instantons as solitons of a five-dimensional theory - instanton particles.

What we have seen above is the standard procedure we will also use below to find supergravity solitons. Let me repeat the salient points:

1. Given a supersymmetric field theory, to find a purely bosonic confguration preserving some fraction of the original supersymmetry, we first of all set to zero the fermionic fields.

2. Then the supersymmetry variations of the bosonic fields are automatically zero (since they tranform into the fermionic fields).

3. Setting to zero the supersymmetry variation of the fermionic fields, one obtains first order equations for the bosonic fields which may have non-trivial solutions for some subset of the original set of supersymmetry parameters.

4. Having solved these equations, one still has to check that the (second order) field equations are satisfied (this is not automatic!).

1.3 What (Numerology and p-Brane Taxonomy)

To obtain a tractable system of equations, we still need to understand what kind of symmmetry conditions we can or should impose prior to imposing the supersymmetry (BPS) condition. To that end it will be extremely helpful to try to anticipate (admittedly with a certain amount of hindsight) what kind of solutions we might hope to find.

What? Differential Forms?

In this section and subsequent sections, it will be extremely useful, in order not to clutter the equations with an infinitude of indices, to occasionally use differential form notation. While there is a whole calculus and theory behind this (which is well worth learning), you may think of this as just a shorthand notation for dealing with antisymmetric tensor fields:

1. A *p-form* A corresponds to an antisymmetric tensor field with p indices,

$$A \Leftrightarrow A_{\mu_1 \dots \mu_p} = A_{[\mu_1 \dots \mu_p]} \quad . \tag{1.15}$$

   In particular, a 0-form is a function (scalar) and a 1-form is a covector field.

2. The *exterior derivative* dA of a p-form A is a $(p+1)$-form and corresponds to the completely antisymmetric curl of A,

$$dA \Leftrightarrow \partial_{[\mu_{p+1}} A_{\mu_1 \dots \mu_p]} \quad . \tag{1.16}$$

   Remember from tensor calculus that no metric or connection is required to define this derivative in a generally covariant way.

3. Given a metric $g_{\mu\nu}$, the *Hodge dual* $*A$ of a p-form in d dimensions is the $(d-p)$-form corresponding to the antisymmetric tensor

$$*A \Leftrightarrow \tilde{A}_{\nu_{p+1} \dots \nu_d} \sim \sqrt{g} \epsilon_{\nu_1 \nu_2 \dots \nu_d} g^{\nu_1 \mu_1} \dots g^{\nu_p \mu_p} A_{\mu_1 \dots \mu_p} \quad . \tag{1.17}$$

   I am not careful with numerical factors here. Note that acting twice with the Hodge operator maps a p-form to a p-form. The Hodge operator is normalized in such a way that $* \circ * = \pm 1$.

It follows from this that $d(dA) = 0$ because second partial derivatives commute and hence vanish upon antisymmetrization. Another consequence is that acting first with with $*$, then with d, then again with $*$, a p-form A is mapped to the $(p-1)$-form corresponding to the covariant divergence of A

$$*d*A \Leftrightarrow \nabla^\mu A_{\mu\mu_1\ldots\mu_{p-1}} \quad . \tag{1.18}$$

Thus an equation of the form $dA = *B$ implies that B has vanishing divergence.

A fundamental property of p-forms is that they are p-dimensional volume elements which can naturally be integrated over p-dimensional spaces V, i.e.

$$\int_V A \tag{1.19}$$

is well defined and generally covariant without having to take recourse to a metric to define something like $\sqrt{g} d^n x$ etc.

If V is the boundary of some $(p+1)$-dimensional region W, $\partial W = V$, *Stokes' Theorem*, generalizing all the classical integral formulae (Green, Stokes, fundamental theorem of calculus, ...), says that

$$\int_W dA = \int_{V=\partial W} A \quad . \tag{1.20}$$

As an example consider the Maxwell equations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.21}$$
$$\partial_{[\mu} F_{\nu\lambda]} = 0 \tag{1.22}$$
$$\nabla^\mu F_{\mu\nu} = J_\nu \quad . \tag{1.23}$$

It is of course clear that the first equation implies the second, and that the third implies that the current J_μ is covariantly conserved,

$$\nabla^\mu F_{\mu\nu} = J_\nu \Rightarrow \nabla^\mu J_\mu = 0 \quad . \tag{1.24}$$

In differential form notation we would write these equations as

$$F = dA \tag{1.25}$$
$$dF = 0 \tag{1.26}$$
$$d*F = J \quad , \tag{1.27}$$

where J is a three-form dual to the current (1-form) $*J$. Once again it is clear that the first implies the second equation which is just an identity, the Bianchi identity, and from the third equation we learn that $dJ = 0$ or that $*J$ has vanishing divergence,

$$d*F = J \Rightarrow dJ = 0 \Leftrightarrow *d*(*J) = 0 \quad . \tag{1.28}$$

The corresponding conserved electric charge can, using Stokes' theorem, be written as a boundary integral over a two-sphere at infinity, so we obtain the integral version of Gauss' law

$$Q_E \equiv \int_{\mathbb{R}^3} J = \int_{S^2_\infty} *F \quad . \tag{1.29}$$

This is really all we will need. You see that using differential form notation saves us from having to write (or read) lots of indices. This will be particularly time-saving when we discuss higher-rank antisymmetric tensor fields like the various RR-fields.

<u>WHAT? NUMEROLOGY?</u>

To understand what kind of solitonic solutions to expect, let us consider the standard example of electrically charged particles versus magnetically charged solitonic monopoles in four space-time dimensions. What we want to stress here is that it is a numerical coincidence that the magnetic dual object of the original electrically charged particle is again a particle in that case. This comes about as follows.

The fact that a particle is electrically charged is reflected in the fact that it couples to the Maxwell gauge field A via a term

$$\int d^4x J^\mu_E(x) A_\mu(x) = \int d\tau A_\mu(x(\tau))\dot{x}^\mu(\tau) \equiv \int_\gamma A \quad , \tag{1.30}$$

where $x^\mu(\tau)$ parametrizes the world line γ of the particle. We see that a point particle couples naturally to a one-form vector potential. This coupling leads to a delta function source term, symbolically

$$d * F = J_E \sim \delta(\gamma) \quad , \tag{1.31}$$

in the Maxwell equations, and this in turn leads to an associated conserved electric charge

$$Q_E = \int_{S^2_\infty} *F \quad , \tag{1.32}$$

where the integral is taken over a two-sphere at spatial infinity enclosing the world line.

Now electric-magnetic duality, $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$, the latter defined in (1.13), corresponds to $F \to *F$ in differential form notation. Thus a hypothetical magnetically dual solitonic particle would carry a magnetic charge

$$Q_M = \int_{S^2_\infty} F \quad , \tag{1.33}$$

and would thus correspond to an object coupling to a gauge field $\tilde{A}$ with curvature $d\tilde{A} = \tilde{F} = *F$. Note that the charge carried by the soliton is indeed a topological quantum number (the monopole number) and not a conserved Noether charge. The former are associated to Bianchi identities ($dF = 0$ identically) and the latter to equations of motion and these two are exchanged under electric-magnetic duality.

The pertinent observation is now that, as $\tilde{A}$ is once again a one-form, the (hypothetical) dual object has to again be a particle. To see that this is indeed a numerical coincidence imagine repeating the same discussion in five space time dimensions, $d = 5$. In that case, $*F$ is a three-form and hence the dual gauge field $\tilde{A}$ would have to be a two-form.

Now what sort of object will couple to a two-form? Of course we already know this: a string. In the standard string theories this is just the coupling to the Neveu-Schwarz (NS-NS) two-form $B_{\mu\nu}$. Thus we see that in five dimensions the magnetic dual to an electrically charged particle is a string. In fact it is a 'monopole string' since, from a five-dimensional perspective, a four-dimensional monopole looks like an object with a (1+1)-dimensional world-volume, i.e. a string. Likewise, in $d = 6$ the magnetic dual of a particle would be a membrane (or 2-brane in modern parlance), an object with a (2+1)-dimensional world volume.

More generally, a $(p+1)$-form $C^{(p+1)}$ provides us with a higher-rank generalization of Maxwell theory. The field strength

$$F^{(p+2)} = dC^{(p+1)} \tag{1.34}$$

is invariant under gauge transformations

$$C^{(p+1)} \to C^{(p+1)} + d\Lambda^{(p)} \quad . \tag{1.35}$$

A gauge invariant Lagrangian d-form is

$$L = F^{(p+2)} * F^{(p+2)} \quad , \tag{1.36}$$

leading to the source-free equations of motion

$$d * F^{(p+2)} = 0 \quad . \tag{1.37}$$

$C^{(p+1)}$ couples naturally to a p-brane, that is an object with a $(p+1)$-diemsional world volume, the coupling taking the form

$$\int_V C^{(p+1)} \quad , \tag{1.38}$$

where V is the world volume of the p-brane. The corresponding charge in d space-time dimensions would then be

$$Q_E = \int_{S_\infty^{d-p-2}} *F^{(p+2)} \quad . \tag{1.39}$$

The magnetically dual object to the p-brane would then be an object coupling to a $(d-p-3)$-form, i.e. a $(d-p-4)$-brane with magnetic charge

$$Q_M = \int_{S_\infty^{p+2}} F^{(p+2)} \quad . \tag{1.40}$$

These equations generalize the Maxwell expressions (1.30,1.32,1.33) from $(d=4, p=0)$ to arbitrary d and p and the numerology behind this reasoning, albeit rather trivial, will play a crucial role in the following.

Solitons and RR Charges

Indeed, let us take seriously this idea in the context of string theory. As we noted above, the ten-dimensional superstring couples to the NS-NS two-form $B_{\mu\nu}$. Thus the curvature is a three-form, its dual a seven-form, corresponding to a dual six-form potential. Thus we conclude that the magnetic dual of a string in ten dimensions, if it exists, is a five-brane! This is the first time that we naturally encounter higher-dimensional extended objects in string theory. Since it couples to a NS-NS field, We will refer to it as the NS 5-brane to avoid confusion with other five-branes that will appear later.

But now we have bitten the bullet. Once we accept the idea that $(p+1)$-form gauge fields are somehow associated with p-branes, i.e. extended objects with a $(p+1)$-dimensional world volume, we see that we might expect a rich spectrum of these p-branes in string theory. Indeed, recall that in addition to the fields from the NS-NS sector, the type IIA and IIB string theories have antisymmetric tensor fields from the Ramond-Ramond (RR) sector.

More precisely, in type IIA we have a RR one-form $C^{(1)}$ and a RR three-form $C^{(3)}$, as well as their (equally fundamental) duals $C^{(7)}$ and $C^{(5)}$. Optimistically we might therefore expect to find, associated with these, a zero-brane (particle) and a two-brane (membrane) respectively, plus their magnetic duals, a six-brane and a four-brane.

I want to stress here (we will come back to that below) that the term 'magnetic dual' is somewhat ambiguous at this point. On the one hand, it can refer to an 'electric' solution simply coupling to a dual RR field (a priori there appears to be nothing in the string theory that prefers a RR p-form to a RR $(6-p)$-form so either one of them can be regarded as 'electric'). As such, it will be a singular solution to the equations of motion following from an action involving this dual $(6-p)$-form RR field.

On the other hand, it can be a 'magnetic' solitonic dual to an 'electric' solution (like the monopole is dual to the electron in certain Yang-Mills Higgs theories), and as such it is a completely non-singular solution to the equations of motion following from the action involving the original 'electric' p-form RR field. Our main interest will be in the 'electric' solutions in the following as it is these that have a nice string theory description in terms of D-branes. However, we will also see that the distinction between 'elementary' electric and 'solitonic' magnetic solutions is somewhat wiped out in string theory since this is a fundamental theory of strings with respect to which both kinds of objects are solitonic. In fact, we will be able to sharpen this statement somewhat at the end of section 2 and then again, in a slightly different way, in section 4.2.

Similarly, returning now to our taxonomy of p-branes, in type IIB there is a RR scalar $C^{(0)} \equiv \chi$ (the axion), a RR two-form $C^{(2)}$, and a RR four-form $C^{(4)}$ with self-dual curvature, $dC^{(4)} = *dC^{(4)}$. Each one of these has its own interesting story.

Let us begin with the RR scalar. According to our numerology, such an object would be associated with a (-1)-brane, i.e. with an object which is point-like in space-time. After Wick rotation, this may be regarded as an instanton. Its magnetic dual (or the object electrically charged with respect to the dual field) would be a seven-brane.

Of course, associated with the RR two-form is a string, but this is clearly not the fundamental type IIB string (which couples to the NS-NS two-form). Thus, in type IIB string theory we actually have (at least) two kinds of strings. In fact, there is a whole $SL(2,\mathbb{Z})$ multiplet of such strings coming from the global $SL(2,\mathbb{R})$ symmetry group of type IIB supergravity under which the two two-forms transform as a doublet - see the lectures by Narain for details. Its magnetic dual would be a 5-brane, again distinct from the NS-NS five-brane mentioned above which is magnetically dual to the fundamental string, not the RR string.

Finally we come to the self-dual RR four-form (where the attribute 'self-dual' of course refers to its curvature). This couples to a three-brane. The interesting thing about this three-brane is that, as a consequence of the self-duality of $dC^{(4)}$, this three-brane is itself self-dual, i.e. equal to its magnetic dual.

If such RR p-branes exist, they are potentially extremely interesting objects. By construction, they are charged under the RR fields. But it is known that there are no RR charged states at all in the perturbative string spectrum (in the vertex operators of perturbative string theory the RR fields only ever enter via their, gauge invariant and hence chargeless, field strengths). Thus not only the magnetic solitonic duals of these RR p-branes (for $-1 \leq p \leq 7$) but already the 'elementary electric' p-branes themselves are of a non-perturbative nature in string theory.

We will see below that all of these RR p-branes (as well as their eleven-dimensional antecedents) can be realized as BPS solitons of the corresponding supergravity theories preserving one half of the supersymmetry. As such these solutions share many of the characteristics of the extremal charged black hole mentioned in the introduction though they also have characteritics unique to string theory.

The key insight, due to Polchinski, making possible many of the recent insights into non-perturbative string theory, was the realization that the new non-perturbative sectors of string theory associated with these solitons can be incorporated naturally into closed string type IIB string theory as certain (Dirichlet) open string subsectors (see the lectures on D-branes by Marco Serone), and hence can be described by methods of *perturbative* open string theory.

1.4 Problems

1. You have thirty seconds to answer the following questions: What is the dual of a string in six dimensions? What is the dual of a membrane in nine dimensions?
2. Verify (1.8) and (1.9) and fill in the details of the argument leading to the Bogomolny'i bound (1.10).

2 Construction of RR p-Brane Supergravity Solitons

The aim in this section is to find the exact supergravity solutions of type II (A and B) string theory realizing the basic RR p-branes whose existence we conjectured before. There are many other interesting solitonic solutions as well, e.g. those of heterotic string theory, or solitons carrying multiple charges etc., but we are not aiming for generality here.

Thus we will make various simplifying assumptions along the way. In particular, we will consider configurations in which only three of the supergravity fields, the metric, the dilaton, and one of the RR fields $C^{(p+1)}$, are non-zero. Their equations of motion can be deduced from a truncated action involving only these fields. We will then impose the maximal symmetry condition compatible with the equations of motion, namely world-volume Poincaré times transverse rotational invariance. This leads to the so-called *p-brane ansatz* for the fields which we will then plug into the (BPS) condition for unbroken supersymmetry.

Once one has satisfied this condition it is a relatively simple matter to solve the remaining equations of motion. Nevertheless, I will not present the details of the calculations (which would require calculating spin connections and curvatures etc.) but only take you through the sequence of steps required to arrive at the solution.

2.1 The Truncated Action

Recall that, in the string or σ-model frame, the bosonic part of the action of type II (A or B) supergravity schematically has the form

$$S_{II} = \int d^{10}x\sqrt{-g_s}\left\{\mathrm{e}^{-2\phi}\left[R(g_s) + 4(d\phi)^2 - \tfrac{1}{12}(dB)^2\right] - \tfrac{1}{2}\sum_p \tfrac{1}{(p+2)!}(F^{(p+2)})^2\right\} \tag{2.1}$$

modulo Chern-Simons terms. Here B is the NS-NS two-form and the sum is over the appropriate (p even/odd for type IIA/B) RR field strengths $F^{(p+2)} = dC^{(p+1)} + \ldots$. Newton's constant is given by the asymptotic value ϕ_0 of the dilaton, and on dimensional grounds one has $G_N \sim g_s^2 \ell_s^8$ with $g_s = \exp\phi_0$ and ℓ_s the string length.

The string frame has the advantage that the RR kinetic terms have their canonical form. However, to study classical solutions to the equations of motion it is occasionally more convenient to go to the canonical or Einstein frame, related to the string frame by

$$g_{MN,s} = \mathrm{e}^{\phi/2} g_{MN,e} \quad . \tag{2.2}$$

In this frame the action takes the form

$$S_{II} = \frac{1}{G_N} \int d^{10}x \sqrt{-g_e} \left[R(g_e) - \tfrac{1}{2}(d\phi)^2 - \tfrac{1}{12}\mathrm{e}^{-\phi}(dB)^2 - \tfrac{1}{2}\sum_p \mathrm{e}^{\frac{(3-p)}{2}\phi} \tfrac{1}{(p+2)!}(F^{(p+2)})^2 \right] \tag{2.3}$$

Note that now the RR field strengths have acquired a p-dependent coupling to the dilaton. In particular, the couplings of the dilaton to the NS and RR three-form field strengths are related by $\phi \to -\phi$.

As mentioned above, we will be interested in classical configurations depending on just a small subset of these fields, namely the metric, the dilaton and one of the RR-fields. It is then consistent to look at the equations of motion following from the truncated action

$$S_{II,p} = \frac{1}{G_N} \int d^{10}x \sqrt{-g_e} \left[R(g_e) - \tfrac{1}{2}(d\phi)^2 - \tfrac{1}{2}\mathrm{e}^{a_p \phi} \tfrac{1}{(p+2)!}(dC^{(p+1)})^2 \right] \tag{2.4}$$

($a_p = (3-p)/2$), provided that the Chern-Simons terms do not contribute. For the solutions given below this can readily be verified explicitly.

Here and in the following, p can take any value $0 \leq p \leq 6$, so we allow for the possibility of working with the dual RR fields C^{7-p} (which, to be consistent with the notation of section 1, we should perhaps rather call $\tilde{C}^{7-p}$).

Thus (2.4) is the action we will be working with henceforth. It gives rise to the equations of motion (indices $L, M, N, \ldots$ run from 0 to 9)

$$\begin{aligned} R_{MN} &= \frac{1}{2}\partial_M \phi \partial_N \phi + S_{MN} && (2.5) \\ S_{MN} &= \frac{1}{2(p+1)!}\mathrm{e}^{a_p \phi} \left(F_{ML_1 \ldots L_{p+1}} F_N^{\;L_1 \ldots L_{p+1}} - \frac{p+1}{8(p+2)} g_{MN} F^2 \right) && (2.6) \\ 0 &= \nabla_M \left(\mathrm{e}^{a_p \phi} F^{ML_1 \ldots L_{p+1}} \right) && (2.7) \\ \Box \phi &= \frac{a_p}{2(p+2)!}\mathrm{e}^{a_p \phi} F^2 && (2.8) \end{aligned}$$

Actually, as mentioned above, when looking for 'electric' solutions to these equations, we expect them to be singular solutions to the equations of motion. Thus, in order to have a consistent solution, we should really supplement the truncated supergravity action by a source term, i.e. a p-brane action given by an integral over the $(p+1)$-dimensional world volume of the brane. This will lead to delta-function like source terms in the equations of motion (2.8). Such an action will roughly have the form

$$S_{source} = T_p(S_{DBI,p} + \int_{V_{p+1}} C^{(p+1)}) \tag{2.9}$$

where DBI means 'Dirac-Born-Infeld', and this action describes, among other things, the embedding of the brane into space-time. Here T_p is the *brane tension*, something like the mass density of the p-brane. In principle, we should of course have written (2.9) as

$$S_{source} = T_p S_{DBI,p} + \rho_p \int_{V_{p+1}} C^{(p+1)} \quad , \tag{2.10}$$

with ρ_p the charge density of the brane. However, then the BPS condition (saturation of the Bogomolny'i bound (mass)$\geq$(charge) (1.10)) will force $T_p = \rho_p$ and we are anticipating this in (2.9).

2.2 The p-Brane Ansatz

As explained in section 1, we are seeking solutions which represent the embedding of the world-volume of a p-brane into space-time. Looking for the simplest solutions possible, i.e. those with the highest degree of symmetry, we will make an ansatz for the metric compatible with Poincaré$_{p+1}$ invariance of the world-volume. We will also require rotational $SO(9-p)$ invariance (isotropy) in the directions transverse to the p-brane.

It is then convenient to split the ten coordinates x^M into longitudinal coordinates $x^\mu, \mu = 0, \ldots, p$ and transverse coordinates $y^m, m = p+1, \ldots, 9$. An ansatz for the metric compatible with the required symmetries is then

$$ds^2 = \mathrm{e}^{2A(r)} d\vec{x}^2 + \mathrm{e}^{2B(r)} d\vec{y}^2 \quad , \tag{2.11}$$

where $d\vec{x}^2$ is the Minkowski metric, $d\vec{y}^2$ is the flat Euclidean metric on the transverse space, $r = \sqrt{\vec{y}^2}$ is the transverse radial coordinate,

$$d\vec{y}^2 = dr^2 + r^2 d\Omega^2_{8-p} \quad , \tag{2.12}$$

and $A(r)$ and $B(r)$ are functions of r to be determined. Coordinates in which the metric takes this form are (for obvious reasons) called *isotropic coordinates*.

The ansatz for the dilaton is simply

$$\phi = \phi(r) \quad . \tag{2.13}$$

For the RR field, the ansatz depends on whether we are interested in what we called above the 'electric' solutions or their solitonic 'magnetic' duals. In the former case, to which the above ansatz with the metric is adapted (for a magnetic solution we would have to replace p by $(6-p)$) the natural ansatz compatible with the symmmetries is

$$C^{(p+1)}_{01\ldots p} = \mathrm{e}^{C(r)} \quad , \tag{2.14}$$

with all other components zero.

For the magnetic solutions, on the other hand, the integral of $dC^{(p+1)}$ over the $(p+2)$-sphere at spatial infinity is required to be non-trivial. Thus an ansatz for $C^{(p+1)}$ would have to exhibit Dirac-string like singularities. It is therefore more convenient to make an ansatz directly in terms of its (globally defined) field strength, and the most convenient choice is to let $F^{(p+2)}$ be proportional to the volume form on S^{p+2}. Thus in this case no arbitrary function $C(r)$ is involved in the ansatz and this simplifies somewhat the analysis. In the following, however, we will primarily focus on the electric solutions.

It would now be possible to calculate the Christoffel symbols (spin connection) and curvature components of the above metric and plug this ansatz into the equations of motion. However, as we are interested in solutions preserving some fraction of the supersymmetry, it will be much more convenient to satisfy this condition first.

2.3 Supersymmetry Invariance: the BPS Condition

As explained above, requiring supersymmetry of the background bosonic configuration provides a shortcut to finding supersymmetric BPS solitons. In the present case it can be shown that the four arbitrary functions A, B, C, ϕ are reduced to one by the requirement that the field configuration represented by the above ansatz preserve some fraction of the supersymmetry. The remaining function is then determined by the field equations. I will not present the details of this calculation, but roughly the strategy is the following:

1. First of all, as we are looking at a purely bosonic configuration, i.e. with zero background fermionic fields, the supersymmetry variations of the bosonic fields (being proportional to the background fermionic fields) are then automatically zero. Thus to check for supersymmetry invariance, we only have to check that there are supersymmetry parameters for which the supervariations of the gravitino and dilatino fields in this bosonic background are zero. Thus we need to know these variations.

2. In the situation we are interested in, the NS-NS field B and all but one of the RR fields $C^{(p+1)}$ are zero, and in that case the transformation rules simplify accordingly. In the string frame, these transformations can be written rather compactly as

$$\begin{aligned} \delta\psi_M &= D_M\eta + \tfrac{(-1)^p}{8(p+2)!}\mathrm{e}^{\phi}\Gamma\bullet F^{(p+2)}\Gamma_M\eta'_{(p)} = 0 \\ \delta\lambda &= \partial_M\phi\Gamma^M\eta + \tfrac{3-p}{4(p+2)!}\mathrm{e}^{\phi}\Gamma\bullet F^{(p+2)}\eta'_{(p)} = 0 \ , \end{aligned} \tag{2.15}$$

where

$$\Gamma\bullet F^{(p+2)} = F_{M_1\ldots M_{p+2}}\Gamma^{M_1\ldots M_{p+2}} \tag{2.16}$$

and $\eta'_{(p)}$ is given by

$$\begin{aligned}
\eta'_{(p)} &= \eta \text{ for } p = 0, 4 \\
\eta'_{(p)} &= \Gamma_{11}\eta \text{ for } p = 2, 6 \\
\eta'_{(p)} &= i\eta \text{ for } p = -1, 3, 7 \\
\eta'_{(p)} &= i\eta^* \text{ for } p = 1, 5
\end{aligned} \tag{2.17}$$

3. Furthermore, the spinor parameter η itself is restricted by the symmetries of the ansatz to be of the form $\eta = \epsilon_{p+1} \otimes \eta_{9-p}(r)$ where ϵ_{p+1} is a constant $SO(p,1)$ spinor and $\eta_{9-p}(r)$ is a spinor of $SO(9-p)$ depending only on the transverse radial coordinate r.

4. Of course, these equations will impose various differential relations among the unknown functions A, B, C and ϕ. Moreover, by the first (Killing spinor) equation, $\eta_{9-p}(r)$ will be related to a constant $SO(9-p)$ spinor η_0. The important points to note are the following:

   (a) For $p \neq 3$, the dilatino equation imposes an algebraic rather than a differential constraint on η, namely
   $$\eta + \Gamma_{\underline{01}\ldots\underline{p}}\eta'_{(p)} = 0 \tag{2.18}$$
   (see also the D-brane lectures where precisely this constraint should have made an appearance) and thus halves the number of unbroken supersymmetries. Here the gamma matrices $\Gamma_{\underline{M}}$ refer to an orthonormal basis.

   [The case $p = 3$ is different. In that case, the dilatino variations simply says that the dilaton is constant, and the algebraic constraint on η arises from the r-component of the gravitino equation.]

   (b) The dilatino equation allows one to express the (derivative of the) dilaton in terms of the (derivative of the) function $C(r)$.

   (c) The Killing spinor equation can be solved in terms of a constant spinor, and the remaining functions $A(r)$ and $B(r)$ can then also be expressed in terms of $C(r)$.

   (d) Demanding that the metric be asymptotically Minkowskian and that the asymptotic value of the dilaton be equal to zero, the differential relationships between C and the other functions can be integrated to give the linear relationships (in the Einstein frame)
   $$\begin{aligned}
   A &= \frac{7-p}{16}C \\
   B &= \frac{-(p+1)}{16}C
   \end{aligned}$$

$$\phi = \frac{p-3}{4}C \ . \tag{2.19}$$

where $\lim_{r\to\infty} C(r) = 0$.

(e) In the string frame one has instead

$$\begin{aligned} A &= \frac{1}{4}C \\ B &= -\frac{1}{4}C \ . \end{aligned} \tag{2.20}$$

(f) If one wants to allow for a non-trivial vev ϕ_0 of the dilaton (but nevertheless retain an asymptotically flat solution), then the above relations are modified to

$$\begin{aligned} A &= \frac{7-p}{16}(C-C_0) \\ B &= \frac{-(p+1)}{16}(C-C_0) \\ \phi &= \phi_0 + \frac{p-3}{4}(C-C_0) \ , \end{aligned} \tag{2.21}$$

where $\lim_{r\to\infty} C(r) = C_0$ and

$$C_0 = \frac{p-3}{4}\phi_0 \ . \tag{2.22}$$

Once again this has its obvious string frame counterpart.

2.4 The type II p-Brane Solutions

While the procedure outlined in the previous subsection sounds fairly involved, in the end it kindly leads to a rather simple expression for the sought-for soliton solution. The key observation is that all of the supergravity field equations are solved provided that $\exp(-C)$ is a harmonic function $H_p(r)$ of the transverse variables. With the above asymptotic condition the solution for C is thus

$$H_p = \mathrm{e}^{-C} = \mathrm{e}^{-C_0} + \frac{Q_p}{r^{7-p}} \ , \tag{2.23}$$

where Q_p is some dimensionful constant.

Actually, if one wants to retain a harmonic function which tends to 1 at infinity, then it is better to define H_p as

$$H_p = \mathrm{e}^{-(C-C_0)} \ . \tag{2.24}$$

Unless explicitly specified to the contrary, we will choose $C_0 = 0$ in the following (we will need $\phi_0 \neq 0$ when discussing the dependence of the string tension T_p on the string coupling constant $g_s = \exp\phi_0$).

Modulo numerical factors (involving the volume of the transverse sphere etc.) which are irrelevant for us, the coupling to the source-terms fixes the constant Q_p to be

$$Q_p = T_p G_N \quad . \tag{2.25}$$

Note that dimensionally this works out: On dimensional grounds, Q_p has to be proportional to ℓ_s^{7-p}, G_N is proportional to $g_s^2 \ell_s^8$, hence T_p has dimension $\ell_s^{-(p+1)}$, precisely right for the mass density of a p-brane.

Thus, putting everything together, the basic electric (or 'elementary') p-brane supergravity soliton solution is (in the Einstein frame)

$$\begin{aligned} ds_e^2 &= H_p^{(p-7)/8} d\vec{x}^2 + H_p^{(p+1)/8} d\vec{y}^2 \\ C_{01\ldots p}^{(p+1)} &= H_p^{-1} - 1 \\ \mathrm{e}^{2\phi} &= H_p^{(3-p)/2} \quad . \end{aligned} \tag{2.26}$$

This solution looks even simpler in the string frame. Using

$$\mathrm{e}^{\phi/2} = H_p^{(3-p)/8} \quad , \tag{2.27}$$

one finds that the metric takes the form

$$ds_s^2 = H_p^{-1/2} d\vec{x}^2 + H_p^{+1/2} d\vec{y}^2 \tag{2.28}$$

while the dilaton and RR field are unchanged.

Of particular interest is the three-brane solution which has a constant dilaton and thus takes the same form in the Einstein and string frames.

It is relatively easy to verify that with this ansatz the supervariations (2.15) vanish provided that (2.18) holds and one has

$$\eta(r) = H_p^{-1/8} \epsilon_{p+1} \otimes \eta_0 \quad . \tag{2.29}$$

Given this solution, it is also straightforward to construct the elementary string soliton, i.e. the configuration electrically charged under the NS two-form B. Indeed, notice that the couplings of the RR and NS two-forms in the type IIB action in the Einstein frame, equation (2.3), only differ by the sign of the dilaton. Thus the string soliton solution in the Einstein frame is

$$\begin{aligned} ds^2 &= H_{p=1}^{-3/4} d\vec{x}^2 + H_{p=1}^{1/4} d\vec{y}^2 \\ \mathrm{e}^{2\phi} &= H_{p=1}^{-1} \\ B_{01} &= H_{p=1}^{-1} - 1 \quad . \end{aligned} \tag{2.30}$$

In particular, therefore, when the fundamental string is weakly coupled, the RR string is strongly coupled and vice-versa. This suggests, albeit rather speculatively, that the strong coupling dynamics of the type IIB string might be described in terms of a weakly coupled RR string theory. See the lectures by Narain for details of this.

The above solutions have the interpretation of a single or multiple brane sitting at the origin $\vec{y} = 0$ of the transverse space. There is a simple generalization of this solution to multiple-center configurations, i.e. to solutions describing parallel but not coincident branes sitting at points $\vec{y}_k$ of the transverse space. These solutions are obtained by replacing the harmonic function H_p of (2.23) by

$$H_p = 1 + \sum_k \frac{Q_p}{|\vec{y} - \vec{y}_k|^{7-p}} \tag{2.31}$$

(this obviously breaks the transverse rotation invariance). Ultimately the reason for why one can superimpose single-brane solutions in such a way is the BPS nature of the solutions which is a sort of 'no force condition'. Here there is an exact cancellation between the gravitational attraction and the repulsive force mediated by the antisymmetric tensor field.

2.5 Electric vs. Magnetic and Elementary vs. Solitonic Solutions

As this may be a confusing point (and it was (is?) certainly confusing to the lecturer), let me come back once again to the issue of solutions electrically charged with respect to the magnetically dual RR fields versus magnetically dual solitons. The former are not solitonic 'magnetic' duals to the 'electric' solutions in the standard sense. In fact, all of the above solutions follow from the 'electric' ansatz for the gauge field. Furthermore, and more importantly, for $p \neq 3$ they actually solve the coupled supergravity-brane bulk-source system with δ-function sources $\delta(r)$. (The case $p = 3$ is special - we will come back to that below.)

On the other hand, there *are* genuinely solitonic solutions as well, solving the source-free equations. These are obtained by replacing p by $(6-p)$ in the metric and the dilaton of (2.26) (so now also $x^\mu, \mu = 0, \ldots, 6-p$ etc.). The electric gauge field ansatz then has to be replaced by a magnetic ansatz which makes the field strength proportional to the volume-form on the transverse $(p+2)$-sphere.

The main difference between an elementary (electric) $(6-p)$-brane and the solitonic $(6-p)$-brane dual to an elementary p-brane is thus the fact that the former is a singular solution of the theory described by the action $S_{II,6-p}$ (i.e. of the action (2.4) with $p \to (6-p)$) while the latter is a completely non-singular solution of the theory descibed by the action $S_{II,p}$.

To be completely explicit about this, let us compare the electric RR p-brane solution with the solitonic magnetic p-brane dual to the elementary electric $(6-p)$-brane. The solution for the former we have derived above, but we will repeat it here for convenience. The string frame metric and dilaton are (2.26,2.28)

$$\begin{aligned} ds_s^2 &= H_p^{-1/2} d\vec{x}^2 + H_p^{+1/2} d\vec{y}^2 \\ \mathrm{e}^{2\phi} &= H_p^{(3-p)/2} \ . \end{aligned} \tag{2.32}$$

For comparison purposes with the solitonic solution, it will be convenient to give directly the RR field strength $F^{(p+2)}$ rather than its potential. By construction it is such that the associated Noether electric charge Q_E is non-zero. Since the equation of motion is (in form notation)

$$d * (\mathrm{e}^{(3-p)\phi/2} F^{(p+2)}) = 0 \ , \tag{2.33}$$

this electric charge is proportional to

$$Q_E = \int_{S_\infty^{8-p}} \mathrm{e}^{(3-p)\phi/2} * F^{(p+2)} \ . \tag{2.34}$$

Thus, in order to have a non-zero electric charge, $F^{(p+2)}$ has to be such that the integrand in the above equation is proportional to the (normalized) volume element Ω_{8-p} on the transverse sphere. We can therefore write the electric solution for $F^{(p+2)}$ (modulo numerical factors) as

$$\mathrm{e}^{(3-p)\phi/2} * F^{(p+2)} = \Omega_{8-p} \ . \tag{2.35}$$

Now let us come to the solitonic p-brane solution, the magnetic dual to the electric $(6-p)$-brane, and as such a solution to the equations of motion following from the truncated action (2.4) for a RR $(6-p)$-form. It has exactly the same metric and dilaton (2.32) as the electric p-brane, and this time, in order to have a non-vanishing magnetic charge Q_M, it is the curvature $F^{(8-p)}$ of the RR field $C^{(7-p)}$ that is proportional to Ω_{8-p}, i.e.

$$F^{(8-p)} = \Omega_{8-p} \ . \tag{2.36}$$

We thus see that if we make the identification

$$F^{(8-p)} = \mathrm{e}^{(3-p)\phi/2} * F^{(p+2)} \ , \tag{2.37}$$

these two objects, although solutions to different equations of motion, represent the same object sitting in space-time. In fact, (2.37) is the natural definition of the dual field as it indeed exchanges equations of motion and Bianchi identities and also maps the truncated action $S_{II,p}$ to the truncated action $S_{II,6-p}$. The distinction between the two is a matter of choice and arises only via the source term in the action, and it is only this which dictates which one should be regarded as elementary and which as solitonic.

We will see later (section 4) that the M-theory picture appears to prefer the solitonic to an electric four-brane and that type IIB $SL(2,\mathbb{Z})$ duality prefers solitonic over elementary RR five-branes. Whether this is trying to tell us something or not, we do not know (there are some other indications, of course, that in spite of a certain amount of p-brane democracy lower-dimensional objects are preferred).

Incidentally, the fact that the RR three-brane is its own dual is reflected in the fact that it is a completely non-singular solution of the equations of motion. This is related to the fact that the RR three-brane is singled out among the RR p-branes by having a constant dilaton.

2.6 Problems

1. Show that the dilatino variation equation in (2.15) is satisfied by the solution (2.26,2.28) for any function H, not necessarily harmonic, provided that the algebraic constraint (2.18) is satisfied. [If you have too much time on your hands, calculate the spin connection and show that the gravitino variation equation is also satisfied for any H provided that additionally (2.29) holds.]

2. Consider a d-dimensional action of the form

$$S = \int d^dx\sqrt{-g}\mathrm{e}^{\alpha\phi}[R(g_{\mu\nu}) + \ldots] \ , \tag{2.38}$$

where ϕ is a scalar field, $g_{\mu\nu}$ is the metric, g its determinant, $R(g_{\mu\nu})$ its Ricci scalar, and α is some real number. Consider a conformal transformation

$$g_{\mu\nu} \to G_{\mu\nu} = \mathrm{e}^{\beta\phi}g_{\mu\nu} \tag{2.39}$$

and, using a scaling argument, determine the value of β as a function of α and d for which the action in terms of $G_{\mu\nu}$ takes the canonical Einstein form

$$S = \int d^dx\sqrt{-G}[R(G_{\mu\nu}) + \ldots] \ , \tag{2.40}$$

Verify that for $d = 10$ you reproduce the relation (2.2) between the Einstein and string frame metrics.

3 Further Properties of RR-Branes

We will now study some properties of the RR p-brane solitons found in the previous section. The main motivation here is to unearth those properties which make it plausible that these solitons are nothing other than the long-distance description of the (microscopic) D-branes.

3.1 Masses and Charges: the BPS Condition

From the experiences with BPS solitons in supersymmetric field theories and the fact that the p-brane solitons constructed above preserve one half of the supersymmetry, one might expect that also the p-brane solitons saturate a (mass) $\geq$ (charge) Bogomolny'i bound of the type (1.10), analogous to the extremality condition for the RN black hole of the introduction. With the minor modification that in the present context of spatially extended solitons it is more appropriate to talk about mass and charge densities rather than the (infinite) mass, so that the 'central' charges actually carry tensor indices, this is indeed the case.

One way of seeing this is to start with the extended supersymmetry algebra of type II supergravity, to derive a Bogomolny'i bound from the positivity of the operator $\{Q, Q\}$ relating the electric or magnetic charges Q_E and Q_M to the mass (density), and show that it is saturated by configurations satisfying (2.15,2.18).

One can calculate directly the charges Q_E and Q_M using (1.39,1.40) (this is easy) and the p-brane mass densities using a generalization of the ADM mass formula of general relativity. This calculation, which will not be undertaken here, shows that the p-brane solitons indeed saturate this bound. None of this should come as a surprise.

However, an interesting (and perhaps at first surprising) fact is that this calculation, if performed in a non-trivial ϕ_0 background, also reveals that, in the string frame, the tension/mass of the RR p-branes behaves as

$$T_p \sim \frac{1}{g_s \ell_s^{p+1}} \tag{3.1}$$

where

$$g_s = \mathrm{e}^{\phi_0} = \mathrm{e}^{\langle \phi \rangle} \tag{3.2}$$

is the dimensionless string coupling constant. We will derive this key relation below.

The important thing to note about (3.1) is that this dependence on the coupling constant differs from standard solitonic effects in field theory which typically go like $1/g^2$, and also from the effects produced by NS-NS solitons, i.e. the NS5-brane, in string theory, which has the expected behaviour

$$T_{NS5} \sim \frac{1}{g_s^2 \ell_s^6} \quad . \tag{3.3}$$

This $1/g_s$-behaviour of RR p-branes is thus genuinely stringy. It is also the smoking gun of open string effects, $1/g_s$-contributions typically arising from disc-diagrams. Thus, even though type II supergravity arises as the low-energy limit of closed string theroy, on the basis of this completely classical calculation alone one might suspect that RR p-branes are related to open strings.

An obvious but interesting consequence of the $1/g_s$ behaviour of the string tension is the following. Even though the tension (mass density) of a RR brane goes to infinity at weak coupling, $g_s \to 0$, its gravitational interaction, governed by

$$G_N T_p \sim g_s \quad , \tag{3.4}$$

goes to zero. This suggests that there ought to exist flat space descriptions of these objects involving open strings. As the tension of these objects goes to infinity as $g_s \to 0$, their dynamics is frozen. And indeed in this limit these are nothing other than the D-branes you have already heard so much about this week.

3.2 The RR p-Brane Tension

As the $1/g_s$ behaviour of the brane tension is such an important property of RR- and D-branes, we want to briefly sketch an argument establishing this behaviour in the supergravity context.

Recall that the RR p-branes are characterized by a harmonic function

$$H_p - 1 \sim \frac{T_p G_N}{r^{7-p}} \quad , \tag{3.5}$$

where T_p is the RR p-brane tension for $g_s = 1$. We will denote this quantity by t_p in the following.

For the purposes of this section we will, in order to keep track of the g_s-dependence, have to allow for a non-zero asymptotic value ϕ_0 of the dilaton. In order to be able to talk about masses and tensions, we need a metric which is asymptotically Minkowskian (and not just some constant multiple of the Minkowski metric) and thus we need to work with the harmonic function (2.24). We see that this amounts to multiplying the numerator $t_p G_N$ by $\exp C_0$ and therefore we find that the string tension in a non-trivial dilaton background is

$$T_p = t_p \mathrm{e}^{C_0} \quad . \tag{3.6}$$

This result can also be obtained from a calculation of the ADM mass density of a RR p-brane.

Recalling the relation

$$C_0 = \frac{p-3}{4}\phi_0 \quad , \tag{3.7}$$

we can rewrite this as

$$T_p = t_p g_s^{(p-3)/4} \quad . \tag{3.8}$$

Now this seems to be somewhat of a disappointment, because it just does not look right. However, we have to remember that we added the source term (2.9) to the bulk supergravity action in the *Einstein frame*. Thus in this frame the RR string tension behaves as $g_s^{-1/2}$, etc. while e.g. the fundamental string tension goes like $g_s^{+1/2}$. To make it explicit that the above tension is the Einstein frame tension, we will denote it by $T_{p,e}$ so that the more precise version of the previous equation reads

$$T_{p,e} = t_p g_s^{(p-3)/4} \quad . \tag{3.9}$$

Now let us check what happens when we go to the string frame. The relevant part of the source action for calculating masses (or tensions) is, of course, the gravitational part of the Dirac-Born-Infeld action, which is schematically of the form

$$S_{DBI,p} \sim \int \sqrt{\det g_{MN}\partial_\mu X^M \partial_\nu X^N} \quad , \tag{3.10}$$

where X^M are the embedding coordinates and μ, ν are world sheet indices. In principle, there could also be some dilaton dependence in this action but its presence would have no effect on the following argument and we will therefore ignore it.

When we transform to the string frame,

$$g_{MN,s} = \mathrm{e}^{\phi/2} g_{MN,e} \quad , \tag{3.11}$$

we find

$$\sqrt{\det g_{MN,s}\partial_\mu X^M \partial_\nu X^N} = \mathrm{e}^{(p+1)\phi/4} \sqrt{\det g_{MN,e}\partial_\mu X^M \partial_\nu X^N} \quad . \tag{3.12}$$

Therefore the tensions in the string and the Einstein frame are related by

$$T_{p,s} = \mathrm{e}^{-(p+1)\phi_0/4} T_{p,e} \quad . \tag{3.13}$$

Putting this together with the above, we find

$$\begin{aligned} T_{p,s} &= \mathrm{e}^{-(p+1)\phi_0/4} T_{p,e} \\ &= g_s^{-(p+1)/4} T_{p,e} \\ &= g_s^{-(p+1)/4} g_s^{(p-3)/4} t_p \\ &= t_p g_s^{-1} \quad . \end{aligned} \tag{3.14}$$

So we have finally arrived at the desired result. Reinstating the correct powers of the string length, we thus have

$$T_{p,s} \sim \frac{1}{g_s \ell_s^{p+1}} \quad . \tag{3.15}$$

The same argument shows that the string tension of the fundamental string in the string frame is independent of g_s and just behaves as $1/\ell_s^2$, as it should.

In particular, therefore, we have

$$G_N T_p = g_s \ell_s^{7-p} \tag{3.16}$$

and the harmonic functions H_p appearing in the RR p-brane solutions have the form

$$H_p = 1 + \frac{c_p N g_s \ell_s^{7-p}}{r^{7-p}} \quad , \tag{3.17}$$

where c_p is some numerical constant and N is the number of p-branes.

3.3 p-Branes and T-Duality

It follows practically from the definition of D-branes or the way they were originally discovered that T-duality relates Dp-branes of IIA/B string theory to D$(p \pm 1)$-branes of type IIB/A string theory. The same can be seen to be true at the level of the classical soliton configurations and as such is another piece of indirect evidence in favour of our identification between these solitons and the D-branes of string theory.

In order to perform a T-duality (in its simplest form) we need an Abelian isometry of the background configuration. In the present case translations along any of the longitudinal x^μ directions will do. Without loss of generality let this be the direction x^p. Then the T-duality rules (in the string frame) specialized to the p-brane background read

$$\begin{aligned} g'_{pp} &= 1/g_{pp} \\ \mathrm{e}^{2\phi'} &= \mathrm{e}^{2\phi}/g_{pp} \\ C'^{(p)}_{0\ldots(p-1)} &= C^{(p+1)}_{0\ldots p} \quad . \end{aligned} \tag{3.18}$$

Applying this to the configuration (2.26,2.28) one sees that manifestly the metric and RR field of the p-brane soliton are mapped to those of the $(p-1)$-brane soliton, so the only thing to check is the dilaton. Here one has

$$\begin{aligned} \mathrm{e}^{2\phi'} &= \mathrm{e}^{2\phi}/g_{pp} = H_p^{1/2} H_p^{(3-p)/2} \\ &= H_p^{(3-(p-1))/2} \quad , \end{aligned} \tag{3.19}$$

precisely the right power for a $(p-1)$-brane. The only thing to note is that H_p is still a harmonic function of the transverse coordinates, albeit independent of the new transverse coordinate x^p. So this will indeed solve the supergravity equations of motion. The only effect of this x^p-independence of the radial transverse coordinate is that the $(p-1)$-brane is in some sense delocalized in (spread out over) the x^p-direction. This may be hard to visualize but is otherwise perfectly acceptable.

It should now also be clear how to perform a T-duality transformation along a transverse direction, namely by running the above procedure backwards: One starts with the p-brane soliton, delocalizes it in, say, the x^{p+1}-direction and then reads the T-duality rules (3.18) from right to left. In that way one maps a p-brane to a $(p+1)$-brane.

3.4 p-Branes and D-Branes: the evidence

At the risk of being repetitive, let us take stock of what we have achieved so far: We have found p-brane like configurations as solutions of the type II (A or B) supergravity equations of motion. These have the following properties:

1. By construction these solutions are electrically or magnetically charged under the appropriate RR fields.
2. Also by construction, these configurations preserve some fraction of the supersymmetry which, for this simple ansatz, turned out to be one half - the remaining supersymmetry (16 supercharges) is the equivalent of $N=1$ in $d=10$ or $N=4$ in $d=4$.
3. Their masses (tensions) have a characteristic $1/g_s$-behaviour.
4. T-duality relates p-branes to $(p \pm 1)$-branes.
5. We had also argued that there should exist flat space descriptions of these objects somehow involving open strings.

All these properties are in perfect analogy with the properties of D-branes discussed as they arise in string theory. This leaves little doubt that p-brane solitons and D-branes provide the long- and short-distance descriptions of one and the same underlying object.

As put so eloquently by Gabriele Ferretti at his 1998 ICTP String School Lectures:

> This is the Duck Proof: If it walks like a duck and talks like a duck, it's a duck!

3.5 Problems

1. Using the rules (3.18), show that T-duality of a p-brane along a delocalized transverse direction compactified on a circle gives a (p+1)-brane.

2. Come up with a less roundabout classical proof of the relation (3.15) establishing that the string-frame tension of the RR branes is proportional to g_s^{-1} and let me know.

4 M-Branes and their NS- and RR-Descendants

4.1 $d = 11$ Supergravity and M-Branes

Although eleven-dimensional supergravity does not arise as the low-energy limit of any perturbative string theory, it has played an important role in recent developments. In particular, it is now commonly believed that the strongly coupled ten-dimensional IIA string is described at low energies by eleven-dimensional supergravity.

Even though the corresponding non-perturbative quantum theory is not known (it is part of the elusive *M-theory*), the eleven-dimensional point of view has proven to be extremely useful and natural in organizing and understanding the plethora of recently unearthed non-perturbative string phenomena.

One aspect of this is that, just as $d = 11$ supergravity is a useful starting point for discussing IIA supergravity in $d = 10$ (I will explain why below), the solitons of eleven-dimensional supergravity (now known as M-branes) provide an economical way of organizing and obtaining the various RR and NS solitons of type IIA supergravity in lower dimensions. It is thus these M-branes which are the subject of this section. We will see later that there is much more to the relation between the ten- and eleven-dimensional theories.

Recall that the bosonic fields of eleven-dimensional supergravity are just the metric and a three-form potential C, and that the bosonic part of the action is schematically

$$S_{11} = \int d^{11}x\sqrt{-g}\left[R - \frac{1}{2.4!}(dC)^2\right] - \int C \wedge dC \wedge dC \quad . \tag{4.1}$$

As before, the Chern-Simons term will play little role in the following, save for a brief appearance in the next paragraph, and thus we will just consider the action consisting of the first two terms. This action looks very much like the truncated actions (2.4) of ten-dimensional supergravity we considered before, with the crucial difference, of course, that here there is no dilaton (which only arises upon Kaluza-Klein compactification to $d = 10$).

The presence of the three-form potential suggests (by hopefully now very familiar arguments) the existence of a singular fundamental two-brane solution and a dual non-singular solitonic five-brane. However, as C appears explicitly in the Chern-Simons term, not just its field strength, it is not possible to dualize C to a six-form potential. So unlike in ten dimensions, here we do not expect to find in addition a fundamental five-brane of a dual action and a corresponding non-singular membrane soliton.

[You might be a worried at this point that a similar argument using the Chern-Simons terms of type II (A or B) supergravity would invalidate the dualization procedure in

$d = 10$ as well. However, it is easy to see that in these theories one can, by a suitable integration by parts of the Chern-Simons terms if necessary, always arrange for any desired RR field to only enter via its field strength. So dualization of that RR field is then possible.]

The M2-brane metric is $(x^\mu, \mu = 0, 1, 2)$

$$ds^2_{M2} = H^{-2/3} d\vec{x}^2 + H^{1/3} d\vec{y}^2 \quad , \tag{4.2}$$

where H is, as before, a harmonic function of the transverse variables y^m, $m = 3, \ldots, 10$, i.e.

$$H = 1 + \frac{Q_2}{r^6} \quad , \tag{4.3}$$

and C is such that the dual of its field strenght is the volume-form on the transverse seven-sphere. This configuration is a singular solution of the supergravity equations coupled to a membrane source, preserves half of the supersymmetries and saturates a Bogomolny'i bound on its electric charge - all this should sound pretty familiar.

The M5-brane metric is $(x^\mu, \mu = 0, \ldots, 5)$

$$ds^2_{M5} = H^{-1/3} d\vec{x}^2 + H^{2/3} d\vec{y}^2 \quad , \tag{4.4}$$

with $H = 1 + Q_5/r^3$. The field strength of C is proportional to the volume form on the transverse four-sphere. This solitonic configuration is a non-singular solution of the source-free equations, breaks half the supersymmetries, and saturates a Bogomolny'i bound on its magnetic charge. Once again, none of this should come as a surprise.

4.2 NS-Branes and RR-Branes from M-Branes

The most elementary observation suggesting an (albeit tentative) relation between type IIA string theory and a quantum theory of eleven-dimensional supergravity is the fact that type IIA supergravity is the dimensional reduction of eleven-dimensional supergravity.

Indeed, given the amount of supersymmetry ($N = 1$ in $d = 11$ corresponds to a Majorana spinor with 32 supercharges, i.e. $N = 2$ in $d = 10$) and the fact that the dimensionally reduced theory is necessarily non-chiral, up to field redefinitions this could hardly be otherwise.

To check this correspondence at the level of the field content, we note that the $d = 11$ metric provides the $d = 10$ metric, the RR one-form $C^{(1)}$ (as the KK gauge field) and the dilaton, while C provides the RR three-form $C^{(3)}$ (when all three indices are in the ten dimensions) and the NS two-form B (when one of the indices is the one corresponding to the eleventh direction). This is precisely the field content of IIA supergravity.

More specifically, using the KK ansatz

$$\begin{aligned} ds^2_{11} &= \mathrm{e}^{-2\phi/3} ds^2_{10} + \mathrm{e}^{4\phi/3}(dx^{10} - C^{(1)})^2 \\ C &= C^{(3)} + B \wedge dx^{10} \quad , \end{aligned} \tag{4.5}$$

the action (4.1) reduces to the IIA supergravity action in the string frame given in (2.1).

Thus reduction of the above M-brane solitons should give rise to solitons of IIA supergravity. This is indeed the case. As for T-duality, in order to perform this reduction, we require an isometry. Thus we can either reduce along a longitudinal world volume direction or we can delocalize the M-brane in a transverse direction and then perform the KK reduction.

In the former case, we obtain a one-brane from the M2-brane and four-brane from the M5-brane. I encourage you to check that this one-brane is the elementary string soliton (2.30) electrically charged under the NS field B, whose metric and dilaton are (in the string frame)

$$\begin{aligned} ds^2 &= H^{-1} d\vec{x}^2 + d\vec{y}^2 \\ \mathrm{e}^{2\phi} &= H^{-1} \quad , \end{aligned} \tag{4.6}$$

where (of course) $H - 1 \sim 1/r^6$. It is also instructive to check that the M5-brane gives rise to the RR four-brane in this way.

In the latter case, the M2-brane reduces to the RR two-brane, and the M5-brane to the NS5-brane, i.e. the solitonic dual to the elementary string soliton. Let us check this explicitly.

In the string frame the metric and dilaton of the NS5-brane are

$$\begin{aligned} ds^2_{NS5} &= d\vec{x}^2 + H d\vec{y}^2 \\ \mathrm{e}^{2\phi} &= H \quad , \end{aligned} \tag{4.7}$$

with $H - 1 \sim 1/r^2$. We want to see that the KK ansatz (4.5) applied to the M5-brane (4.4) reproduces this solution. To that end, we assemble the ingredients (4.7) into the KK ansatz (4.5) to find

$$\begin{aligned} ds^2_{11} &= \mathrm{e}^{-2\phi/3} ds^2_{10} + \mathrm{e}^{4\phi/3}(dx^{10})^2 \\ &= H^{-1/3} ds^2_{NS5} + H^{2/3}(dx^{10})^2 \\ &= H^{-1/3} d\vec{x}^2 + H^{2/3}(d\vec{y}^2 + (dx^{10})^2) \quad . \end{aligned} \tag{4.8}$$

This is precisely the M5-brane metric (4.4), delocalized in the x^{10}-direction (so that $H - 1 \sim 1/|\vec{y}|^2$ rather than $\sim 1/r^3$).

It is perhaps worth making a comment about the relation between the world volume theories one obtains in this way. As you know by now, the low energy effective world volume dynamics of D-branes are described by the dimensional reduction of $N = 1$ $d = 10$ Maxwell theory to $d = (p+1)$. In particular, the bosonic field content consists of a gauge field and $(9-p)$ scalars. The M2-brane world volume action, on the other hand, is described by a scalar multiplet with 8 scalars (for the transverse directions). So how can dimensional reduction of the M2-brane world volume theory give the D2-brane action? The answer is that after dimensional reduction on a circle, one of the M2 scalars is compact and can be dualized to a $(2+1)$-dimensional gauge field, giving the desired field content of one gauge field plus seven scalars. More poetically, one could turn this around and say that the D2-brane 'knows' about the eleventh dimension by dualizing the gauge field. I do not know how to generalize this argument to the non-Abelian case.

This interpretation is also consistent with the identification, to be discussed below, of some quantum theory of eleven-dimensional supergravity with the strong coupling limit of IIA string theory on the one hand, and the fact that the D2 worldvolume theory is expected to have a strong coupling fixed point with $SO(8)$ R-symmetry on the other, as this symmetry appears naturally as the manifest transverse rotation symmetry of the M2-brane.

The corresponding statements for the M5-brane are less well understood and involve some rather exotic phenomena, even by string theory standards.

In any case, we see that from the M2 and M5 brane we can get the following objects in type IIA string theory: the fundamental NS1 string and its NS5-brane dual, and the (D or RR) 2-brane and its D4-brane dual. Missing are the D0-branes and the dual D6-branes. The former will be the main subject of the next section.

Understanding the latter goes somewhat beyond what we are doing here. Suffice it to say that they arise by uplifting a four-dimensional gravitational instanton, the Taub-NUT metric, to a solution of eleven-dimensional supergravity where it describes (there are now 4 transverse and 6+1 longitudinal directions) a purely gravitational soliton, the so-called TN6-brane. The Taub-NUT metric has one circular direction, and identifying this with a circle in the eleventh direction, one obtains the D6-brane (while reducing along a worldvolume direction one obtains a ten-dimensional TN5-brane).

The IIB D-branes now follow from the IIA branes by T-duality or by compactifying M-theory on a two-torus.

From this eleven-dimensional point of view, which is asymmetric as far as elementary electric and their solitonic magnetic duals are concerned, because the three-form gauge field C cannot be dualized, one sees that M-theory naturally gives the RR p-branes for $p < 3$ and the magnetic duals for $p > 3$. Indeed, take e.g. the M5-brane reduction to a D4-brane. The solitonic M5-brane solution is such that the curvature of the three-form potential C is proportional to the volume-element on the transverse four-sphere. Hence, upon reduction to a four-brane along a world-volume direction, the four-brane will also be magnetically charged. It is therefore not the elementary electric four-brane but actually the magnetic solitonic dual to the RR two-brane.

In this context, let me also mention that the D5-brane that appears naturally in string theory is also the solitonic five-brane. In fact, type IIB string theory has naturally a fundamental NS1 string solution coupling to the NS two-form $B^{(2)}$, and a solitonic NS5-brane. S-duality maps $B^{(2)}$ to $C^{(2)}$ and therefore naturally the NS5-brane to the solitonic D5-brane. The point here is that manifest S-duality requires an action involving $C^{(2)}$ rather than its dual $C^{(6)}$. However, once again, as explained in section 2, this distinction is rather artificial unless one specifies explicitly (via source terms) which object is to be treated as elementary.

4.3 M-Theory as the Strong Coupling Limit of IIA String Theory

Sometimes innocent looking small observations can have far-reaching and amazing consequences. Here is one of them. Let us take another look at the KK ansatz (4.5). We see that there is a relationship between the dilaton ϕ and the (10,10) component of the metric,

$$G_{10,10} = \mathrm{e}^{4\phi/3} \ , \tag{4.9}$$

which translates into the relation

$$R_{10}^3 = g_s^2 \tag{4.10}$$

between the ten-dimensional string coupling constant and the size of the eleventh dimension, measured in eleven-dimensional Planck units. To make this more manifest, we write this relation as

$$R_{10}^3 = g_s^2 \ell_P^3 \ , \tag{4.11}$$

with ℓ_P the eleven-dimensional Planck scale.

We can also read off from the first part of the KK metric that there is a relation between the eleven-dimensional Planck scale ℓ_P and the ten-dimensional string scale ℓ_s, namely

$$\ell_P^2 = g_s^{2/3} \ell_s^2 \tag{4.12}$$

or

$$\ell_P^3 = g_s \ell_s^3 \ . \tag{4.13}$$

Combining these two relations, we find that the relation between the radius of the eleventh dimension, measured in string units, and the string coupling, is

$$R_{10} = \ell_s g_s \quad . \tag{4.14}$$

We see that small g_s corresponds to a small R_{10}, consistent with the fact that perturbative IIA string theory is a ten-dimensional theory. However, taking the above identification seriously suggests that as one increases the ten-dimensional string coupling an eleventh dimension opens up.

The boldest conjecture one could make at this point is that the strong coupling limit of type IIA string theory is a consistent Lorentz-invariant eleven-dimensional quantum theory whose low-energy limit is eleven-dimensional supergravity and which reproduces the IIA string theory upon compactification on a circle. This hypothetical theory, which cannot be a string theory, perhaps a theory of membranes, but perhaps something altogether different, is known as *M-Theory*.

Is there any evidence in favour of this rather bizarre claim? In fact, there is. The most elementary is the following consistency check. Consider the M2-brane with tension

$$T_{M2} = \frac{1}{\ell_P^3} \quad . \tag{4.15}$$

If we wrap this M2-brane over the eleventh direction, we should find the ten-dimensional fundamental string BPS soliton. The tension of this string is obtained by integrating over the circle and (modulo factors of 2π, which we have consistently ignored in the entire discussion anyway) one finds

$$T_{M2} \to \frac{R_{10}}{\ell_P^3} = \frac{\ell_s g_s}{\ell_s^3 g_s} = \frac{1}{\ell_s^2} = T_{F1} \quad , \tag{4.16}$$

which is, as it should be, the tension of the fundamental or NS string. But now that one of the BPS states has been matched, the others should follow as well. Consider therefore reducing the M2-brane along a transverse direction. In that case we should find the D2-brane, and indeed we see that, thanks to (4.13), the M2-brane tension turns into that of the D2-brane,

$$T_{M2} = \frac{1}{\ell_P^3} \to \frac{1}{g_s \ell_s^3} = T_{D2} \quad . \tag{4.17}$$

Of course, in the above, the power of ℓ_s that one finds in the calculation is guaranteed to come out correctly on dimensional grounds, but the check is on the appearance of the correct power of the dimensionless string coupling constant g_s.

While the above is reassuring, it is essentially guaranteed by supersymmetry and the relation between the supersymmetry algebras and BPS equations in ten and eleven dimensions. So this does not really provide a non-trivial check on the claim made above that the type IIA sting theory is really secretly an eleven-dimensional theory which only appears ten-dimensional at very weak coupling, i.e. in perturbation theory.

But we can go further than that. So far we have matched the massless modes of the two theories as well as the tension of the M2-brane with those of the corresponding BPS objects (D2,F1) in ten dimensions. However, if the eleventh dimension is 'real', then one should also find the massive KK modes in type IIA string theory. So where are these massive modes? From the usual KK scenario we know that these are precisely the modes charged under the KK gauge field and that their mass spectrum has the characteristic form

$$m_{nKK}^2 = \frac{n^2}{R_{10}^2} \quad , \qquad n \in \mathbb{Z} \quad . \tag{4.18}$$

On the other hand, from the KK ansatz (4.5) and the discussion preceding it, we know that the KK gauge field is to be identified with the RR one-form $C^{(1)}$ of IIA string theory. The objects charged under this gauge field are, of course, D0-branes. So the eleven-dimensional origin of the D0-brane, which had been missing so far, is plausibly a plane wave with momentum in the eleventh direction (momentum modes in the other directions are also ten-dimensional BPS states - they are of course T-dual to the fundamental winding modes and the reason we have not discussed them here is that they do not carry RR charge).

A single D0-brane has, as we have seen, a mass (tension)

$$T_{D0} \equiv T_{p=0} = \frac{1}{g_s \ell_s} \quad . \tag{4.19}$$

We know that there is no binding energy between D0-branes, because they are half-BPS and we can superimpose D0-branes at different positions. Thus a bound state of n D0-branes would have a mass squared

$$m_{nD0}^2 = \frac{n^2}{\ell_s^2 g_s^2} \quad . \tag{4.20}$$

But thanks to the fundamental relation (4.14) this mass formula agrees precisely with the KK expression (4.18),

$$R_{10} = \ell_s g_s \Rightarrow m_{nKK}^2 = m_{nD0}^2 \quad . \tag{4.21}$$

Once again, the matching of the D0-brane tension with the KK momentum is more or less implied by supersymmetry. However, something non-trivial remains to be checked to obtain a complete matching of the spectra. From the KK point of view, there is one, and only one, mode for each integer n. Thus what needs to be established is that there is a unique bound state of n D0-branes for each n.

The framework within which issues regarding bound states of D-branes are to be addressed is the dynamics of the world-volume supersymmetric gauge theories on the D-branes. In the present case this becomes a very subtle question about bound state wave functions in the D0-brane worldline $U(n)$ supersymmetric quantum mechanics - subtle because the bound states are marginal. To the best of my knowledge, while existence and uniqueness of the bound state is generally believed to be true for arbitrary n, and there is plenty of analytical and numerical evidence in favour of this claim, so far it has been established rigorously only for $n = 2$.

Many other impressive quantitative checks have been performed on the conjecture that the strong coupling limit of type IIA string theory is a quantum theory of eleven-dimensional supergravity and its interconnection with other string dualities. The M-theoretic point of view has also been shown to be enormously powerful in analyzing the strong coupling behaviour of world-volume gauge theories of D-branes, leading e.g. to a completely different derivation of the Seiberg-Witten solution of $N = 2$ supersymmetric gauge theories.

By now there is overwhelming circumstantial evidence in favour of this M-theory conjecture and the entire conjectural web of string theory - M-theory dualities appears to be perfectly consistent. I refer you to the more advanced among the reviews cited in section 7 for more information on these matters.

4.4 Problems

1. Starting with the M5-brane tension
$$T_{M5} = \frac{1}{\ell_P^6} \quad , \tag{4.22}$$
show that one correctly finds either the tension (3.3) of the NS5-brane or that of the D4-brane by reducing the M5-brane along a transverse direction or wrapping it over a circle with radius $\ell_s g_s$.

2. Use the Kaluza-Klein ansatz
$$ds_{11}^2 = \mathrm{e}^{-2\phi/3} ds_{10}^2 + \mathrm{e}^{4\phi/3}(dx^{10} - C^{(1)})^2 \tag{4.23}$$
to show that Kaluza-Klein reduction of the M5-brane metric
$$ds_{M5}^2 = H^{-1/3} d\vec{x}^2 + H^{2/3} d\vec{y}^2 \quad , \tag{4.24}$$
gives the string frame RR four-brane metric
$$\begin{aligned} ds_s^2 &= H^{-1/2} d\vec{x}^2 + H^{+1/2} d\vec{y}^2 \\ \mathrm{e}^{2\phi} &= H^{-1/2} \quad . \end{aligned} \tag{4.25}$$

3. Show explicitly that the M2- and M5-brane metrics (4.2,4.4), with C as indicated, are solutions of the equations of motion of eleven-dimensional supergravity.

5 Intersecting branes and Black holes[2]

In this section we will consider ways in which we can use the brane solutions already found to produce more complicated intersecting brane configurations that preserve fewer supersymmetries. As such these intersecting configurations are of interest for the solitonic solutions to Heterotic and Type I string theories for which there is only half of the supersymmetry of the type IIA/B and M-theory discussed up until now. They also are of major importance for the construction of lower dimensional solutions that have the structure of black holes with finite horizon area at extremality (like the RN black hole).

5.1 Charges supergravity and intersections - the rules

To determine the possible consistency of intersections consider the case of IIB string theory (the consistency conditions for type IIA and M-theory can be derived by a similar construction). In IIB string theory the fields include a 4-form gauge potential C^+ with a self dual 5-form field strength D^+. If we impose the self-duality condition on D^+ only after varying the action then the action must include the Chern-Simons term

$$C^+ \wedge H \wedge H', \tag{5.1}$$

where $H = dB$ and $H' = dB'$ are the field strengths corresponding to fundamental and RR strings.

With the Chern-Simons term, the B equation becomes

$$d(*H - D^+ \wedge B') = 0. \tag{5.2}$$

Of course the equation without CS term was simply $d * H = 0$ which was necessary for the integral of $*H$ to be a well-defined string charge. We see now that it is not the case and in fact the charge must now be the integral of $*H - D^+ \wedge B'$. Let us see the consequence of this extra term for the case in which the string has an endpoint on another brane.

A long way from the intersection the charge of the string will be to a very good approximation given by

$$Q_{F1} = \int_{S^7} *H. \tag{5.3}$$

As the integration sphere S^7 moves towards the endpoint of the string the field due to the presence of the other brane will become stronger and the approximate equality

[2] This section has been contributed by Martin O'Loughlin (SISSA)

will no longer be true. However, if one decreases the radius of the seven sphere as it approaches the intersection then the value of this integral will remain Q_{F1}. When the sphere arrives at the endpoint of the string and point of intersection with the other brane it will have almost zero radius and can then be deformed into $S^5 \times S^2$. Now the integral corresponding to the charge gets approximately all contribution from $-D^+ \wedge B \wedge B'$,

$$Q_{F1} = -\int_{S^5} D^+ \times \int_{S^2} B'. \tag{5.4}$$

The first part of this expression is clearly RR 3-brane charge and thus we can identify the brane intersected by the string in this case to be a D3-brane. The second part of this expression represents the charge of the string endpoint, as seen from within the world-volume of the 3-brane. Indeed the S^2 integration surface is inside the 3-brane world-volume and surrounds the point of intersection. Thus we see that in IIB string theory a fundamental string may end on a D3-brane soliton.

In the above discussion the roles played by B and B' are identical and thus the same argument can be followed through for a D1-brane (D-string) that ends on a D3-brane. Additionally by rewriting the CS term in such a way that the equation of motion includes the field strength of the D5-brane we can show similarly that the string can end on a D5-brane and considering the modified C^+ equation of motion we can show that a D3-brane can end on a D5-brane.

Intersections allowed by the CS-term are supersymmetric, although the presence of an additional brane in general breaks more supersymmetry. One can directly discover the amount of supersymmetry preserved by writing down the soliton solution corresponding to the intersection and then counting the number of conserved spinors. As noted above the same construction can be carried also in the case of IIA string theory with CS-term and in M-theory with CS-term.

Immediately from these consistency conditions we can generate many other consistent configurations using T-duality. A simple example is to consider IIA string theory with a D0-brane inside a D4-brane. This can be related by a T-duality in two of the directions parallel to the D4-brane to a D-string intersecting a D3-brane in a point (which is the intersection for which we have explicitly derived the consistency condition). This can further be related by a T-duality along the D-string and a T-duality transverse to the D3-brane after which we find in IIB theory a D-string inside a D5-brane. This particular configuration is important for the study of black holes in string theory as we will soon see. First however we need to understand how to construct such intersections for the corresponding solitons in type IIA supergravity.

In general for the p-brane solutions in supergravity we found that the volume scaling functions that appear in the metric are harmonic functions. Furthermore the equation for such functions is linear and so linear combinations of harmonic functions are clearly also harmonic. In short this means that we can simply superpose the harmonic functions corresponding to a pair of intersecting branes to easily find the metric of the intersection. For instance the intersection of a D1 and a D3 - brane mentioned above and motivated by using duality transformations is denoted by (x for brane coordinates and $-$ for coordinates transverse to the brane),

$$\begin{array}{c|ccccccccc} x & x & - & - & - & - & - & - & - & - \\ x & - & x & x & x & - & - & - & - & - \end{array}$$

with corresponding metric,

$$\begin{aligned} ds^2 &= H_1(r)^{-\frac{1}{2}} H_3(r)^{-\frac{1}{2}}(-dt^2) \\ &+ H_1(r)^{-\frac{1}{2}} H_3(r)^{\frac{1}{2}} dx_1^2 + H_1(r)^{\frac{1}{2}} H_3(r)^{-\frac{1}{2}}(dx_2^2 + \cdots + dx_4^2) \\ &+ H_1(r)^{\frac{1}{2}} H_3(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_4^2). \end{aligned} \tag{5.5}$$

Both H_1 and H_3 are harmonic functions in the five dimensions transverse to both the D1 and D3 branes,

$$H_1(r) = 1 + \frac{Q_1}{r^3} \qquad H_3(r) = 1 + \frac{Q_3}{r^3}. \tag{5.6}$$

We will denote this configuration D1$\perp$D3.

Notice that all functions depend only upon the directions transverse to both branes indicating that to construct these solutions we need to smear the branes. The presence of the two different branes is represented by the harmonic functions for each brane present and the power of the harmonic function that multiplies the metric transverse to and perpendicular to the corresponding brane.

Another configuration that will be of much interest to us in the following is that corresponding to a D1-brane living inside a D5-brane. The configuration D1$\|$D5 is represented by the following table,

$$\begin{array}{c|ccccccccc} x & x & - & - & - & - & - & - & - & - \\ x & x & x & x & x & x & - & - & - & - \end{array}$$

and has metric

$$ds^2 = K_1(r)^{-\frac{1}{2}} K_5(r)^{-\frac{1}{2}}(-dt^2 + dx_1^2)$$

$$
\begin{aligned}
&+ \quad K_1(r)^{\frac{1}{2}} K_5(r)^{-\frac{1}{2}} (dx_2^2 + \cdots + dx_5^2) \\
&+ \quad K_1(r)^{\frac{1}{2}} K_5(r)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_3^2),
\end{aligned} \tag{5.7}
$$

where now the functions K are harmonic in the four transverse directions, i.e.

$$
K_1(r) = 1 + \frac{Q_1}{r^2} \qquad K_5(r) = 1 + \frac{Q_5}{r^2}. \tag{5.8}
$$

One may complain that these two configurations cannot be simply related by T-duality as claimed in the previous subsection. In fact to relate these solutions via T-duality the D1 $\perp$ D3 configuration needs to be compactified on a circle transverse to both branes and thus the harmonic functions need to averaged over this additional circle (as already discussed in section 3.3). After this averaging they become harmonic functions in only four coordinates and can then easily be related via T-duality to the D1 $\|$ D5 configuration.

5.3 p-branes and Black holes

One of the interesting and physically revealing applications of brane solitons is to the study of black hole configurations in the supergravity theories arising from string theory. As discussed in the introduction black holes form a special class of brane solutions. In four-dimensional Einstein gravity a black-hole is a point-like object in space that is described by a one-dimensional world-line in space-time, as such the black hole is a zero-brane.

The prototypical example that is of relevance for our discussion, is the Reissner-Nordstrom black hole. In its non-extremal form, $M > Q$, its properties are similar to those of the standard Schwarzschild black hole, whereas for mass almost equal to but greater than the charge, the black hole has special properties related to the fact that when $M = Q$, the black hole is BPS and stable. In this case there is a unique horizon of radius Q. The thermodynamic entropy of the black hole is related to the horizon area via $\frac{1}{4}A_{horizon}$. This general pattern of properties holds true for a very large class of charged p-branes.

One of the first interesting calculations to be done for Dp-brane black holes was an entropy counting carried out by counting the number of string states that are present for a D1 $\|$ D5 system and then comparing it to the geometric entropy of the corresponding soliton. If one considers all the coordinates of the branes to be compactified reducing the configuration to a point-like object in five dimensions, we see that this object is indeed a black hole. However we need to modify it a little so that the horizon area is finite thus giving a non-zero number to be compared to the string theory calculation for a non-trivial check on the string theory picture of black holes.

In fact if we compactify the four coordinates of the D5-brane that are transverse to the D1-brane, then we find that the horizon of the corresponding five dimensional black hole has a radius proportional to the radius of the D1-brane coordinate. In particular as this radius goes also to zero, the horizon area and consequently the black hole entropy goes to zero.

The trick that saves us is adding momentum along the D1-brane in such a way that supersymmetry is still preserved. Doing this we find for the five-dimensional black hole metric,

$$\begin{aligned} ds_5^2 &= -(K_1(r)K_5(r)(1+L(r))^{-\frac{2}{3}}dt^2 \\ &+ (K_1(r)K_5(r)(1+L(r))^{\frac{1}{3}}(dr^2+r^2d\Omega_3^2), \end{aligned} \tag{5.9}$$

with K_1 and K_5 as above and $L(r) = \frac{r_m^2}{r^2}$. We must keep in mind of course that this solution will be valid provided none of its characteristic scales are close to the string scale ℓ_s.

Calculating the charges we find the number of branes involved in this configuration to be,

$$N_1 = \frac{r_1^2 V}{g_s \ell_s^6}, \quad N_5 = \frac{r_5^2}{g_s \ell_s^2}, \quad N_m = \frac{r_m^2 R^2 V}{g_s^2 \ell_s^8}. \tag{5.10}$$

The horizon is at $r=0$ and therefore the entropy is

$$S_{1\|5} = \frac{A(r)|_{r=0}}{4G_5} \tag{5.11}$$

where $A(r)$ is the area of the S^3 at radius r. The answer is finite and proportional to the square roots of the brane numbers;

$$S_{1\|5} = 2\pi\sqrt{N_1 N_5 N_m}. \tag{5.12}$$

The comparison of this result for the entropy to the microscopic counting via string theory using D-brane techniques, can be found in the original article of Strominger and Vafa (Physics Letters B379, (1996), 99; `hep-th/9601029`) or in one of the various reviews mentioned in the final chapter of these notes. In particular the notes of Peet cover in considerable detail many of the topics discussed in these lecture notes.

6 Branes and AdS Space-Times

In a certain limit, some of the brane configurations we have discussed above asymptote to maximally supersymmetric solutions of supergravity of the form $AdS \times S$ where AdS is anti-de Sitter space and S is a sphere. This is one of the ingredients of the *AdS/CFT Correspondence* to be discussed in the lectures by Massimo Bianchi. Here I provide some background information.

6.1 AdS Coordinate Systems and Metrics

By definition, anti-de Sitter space-time is a space-time of constant negative curvature. We wil denote the $(d+1)$-dimensional anti-de Sitter space-time by AdS_{d+1}. Constant curvature means that its curvature tensor can be expressed in terms of the metric $G_{MN}, M, N = 0, \ldots, d$ as

$$R_{KLMN} = \frac{R}{d(d+1)}(G_{KM}G_{LN} - G_{KN}G_{LM}) \ . \tag{6.1}$$

We will write the constant negative scalar curvature (Ricci scalar) R as

$$R = -\frac{d(d+1)}{R^2_{AdS}} \ , \tag{6.2}$$

where R_{AdS} is the *curvature radius* of AdS_{d+1}.

AdS_{d+1} can be thought of as the homogeneous space

$$AdS_{d+1} = SO(d,2)/SO(d,1) \ . \tag{6.3}$$

A convenient way of realizing AdS_{d+1} is as the hypersurface

$$(X^0)^2 - \delta_{ij}X^iX^j + (X^{d+1})^2 = R^2_{AdS} \tag{6.4}$$

$(i,j = 1,\ldots,d)$ in an ambient flat space $\mathbb{R}^{2,d}$ with signature $(2,d)$ and the metric induced from the flat metric

$$ds^2_{d+2} = -(dX^0)^2 + \delta_{ij}dX^idX^j - (dX^{d+1})^2 \ . \tag{6.5}$$

Note that this space has closed time-like circles. This can be seen by writing the defining equation as

$$(X^0)^2 + (X^{d+1})^2 = R^2_{AdS} + \delta_{ij}X^iX^j \ , \tag{6.6}$$

which defines a circle in the time-like plane spanned by the coordinates X^0 and X^{d+1}. One usually passes to the universal covering space, replacing this circle by a time-like real line, and this will be done implicitly in the following.

By construction, AdS_{d+1} is maximally symmetric, with isometry group $SO(d,2)$, as this is the invariance group of the defining equation (6.4).

There are several different useful coordinate systems for AdS_{d+1}, exhibiting different slicings of AdS and making manifest different subgroups of the isometry group. These are all easily derived from the above model for AdS. Here are some of them:

Global Coordinates

Introduce a radial coordinate by

$$r^2 = \delta_{ij} X^i X^j \quad , \tag{6.7}$$

where $i,j = 1,\dots,d$. Then (6.4) becomes

$$(X^0)^2 + (X^{d+1})^2 = R_{AdS}^2 + r^2 \quad . \tag{6.8}$$

This can be solved by setting

$$\begin{aligned} X^0 &= (R_{AdS}^2 + r^2)^{1/2} \sin \frac{t}{R_{AdS}} \\ X^{d+1} &= (R_{AdS}^2 + r^2)^{1/2} \cos \frac{t}{R_{AdS}} \quad . \end{aligned} \tag{6.9}$$

Introducing polar coordinates for the X^i, $X^i = rn^i$, with $\delta_{ij} n^i n^j = 1$, one then finds the metric

$$ds^2 = (1 + \frac{r^2}{R_{AdS}^2})^{-1} dr^2 - (1 + \frac{r^2}{R_{AdS}^2}) dt^2 + r^2 d\Omega_{d-1}^2 \quad . \tag{6.10}$$

In these coordinates the slices of constant r have the form $S^{d-1} \times \mathbb{R}$. See Problem 1 in section 6.3 for an alternative form of the metric in global coordinates.

Poincaré Coordinates

As we will see, this system of coordinates arises naturally in the context of supergravity p-branes. The coordinates are a radial coordinate z and 'world-volume' coordinates $x^\mu = (t, x^a, a = 1,\dots,d-1)$. These are related to the X's by

$$\begin{aligned} X^0 &= R_{AdS} \frac{t}{z} \\ X^a &= R_{AdS} \frac{x^a}{z} \\ X^{d+1} - X^d &= R_{AdS} \frac{1}{z} \\ X^{d+1} + X^d &= R_{AdS} (z + \frac{1}{z}(\vec{x}^2 - t^2) \quad . \end{aligned} \tag{6.11}$$

In terms of these coordinates the metric takes the particularly simple form

$$ds^2 = \frac{R_{AdS}^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \ , \tag{6.12}$$

making manifest the invariance of the metric under the Lorentz subgroup $SO(d,1)$ of the isometry group $SO(d,2)$. In terms of the new radial coordinate $r = R_{AdS}^2/z$, this metric becomes

$$ds^2 = \frac{R_{AdS}^2}{r^2}dr^2 + \frac{r^2}{R_{AdS}^2}\eta_{\mu\nu}dx^\mu dx^\nu \ . \tag{6.13}$$

FRW Coordinates

AdS also arises as a special solution to the Friedmann-Robertson-Walker equations. To obtain the AdS metric in FRW form, we set

$$\begin{aligned} X^0 &= R_{AdS}\sin\frac{t}{R_{AdS}} \\ X^{d+1} &= \cos\frac{t}{R_{AdS}}(R_{AdS}^2 + r^2)^{1/2} \\ X^i &= r\cos\frac{t}{R_{AdS}}n^i \end{aligned} \tag{6.14}$$

to find

$$ds^2 = -dt^2 + \cos^2\frac{t}{R_{AdS}}[(1 + \frac{r^2}{R_{AdS}^2})^{-1}dr^2 + r^2 d\Omega_{d-1}^2] \ . \tag{6.15}$$

The spacelike slices of constant t are hyperboloids, and the timelike slices of constant r are, as for global AdS, of the form $S^{d-1}\times\mathbb{R}$, albeit with a time-dependent radius of the sphere.

6.2 Near-Horizon Limits of Non-dilatonic Branes

In this section we will take a look at a nice (but at first sight perhaps rather insignificant) property of the so-called non-dilatonic branes we have encountered so-far, namely the M2- and M5-branes (there is no dilaton in eleven-dimensional supergravity) and the RR three-brane (for which the dilaton is, as we have seen, constant).

The metrics are (cf. (2.28,4.2,4.4))

D3 $\quad ds_{D3}^2 = H_3(r)^{-1/2}d\vec{x}^2 + H_3(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) \quad (6.16)$

M2 $\quad ds_{M2}^2 = H_2(r)^{-2/3}d\vec{x}^2 + H_2(r)^{1/3}(dr^2 + r^2 d\Omega_7^2) \quad (6.17)$

M5 $\quad ds_{M5}^2 = H_5(r)^{-1/3}d\vec{x}^2 + H_5(r)^{2/3}(dr^2 + r^2 d\Omega_4^2) \quad (6.18)$

where $H_p(r)$ is in each case a harmonic function of the transverse radial variable $r = \sqrt{\vec{y}^2}$. The choice for $H_p(r)$ we made in sections 2 and 4, namely

$$\begin{aligned} H_3(r) &= 1+\frac{Q_3}{r^4} && (6.19)\\ H_2(r) &= 1+\frac{Q_2}{r^6} && (6.20)\\ H_5(r) &= 1+\frac{Q_5}{r^3} && (6.21)\end{aligned}$$

was dictated by the requirement of asymptotic flatness. However, one also obtains a (non-asymptotically flat) solution of the field equations if one drops the constant part in H_p, i.e. if one replaces

$$H_p(r) \rightarrow F_p(r) = H_p(r) - 1 \ . \tag{6.22}$$

There are several different ways of looking at this operation of replacing $H_p(r)$ by $F_p(r)$:

1. One can think of the solution that one obtains in this way as the solution describing a stack of a large number $Q_p \rightarrow \infty$ of branes (in this limit one can drop the constant part of H_p).

2. Alternatively, one can think of this solution as describing the geometry of the brane very close to the horizon of the solution, located at $r = 0$ (hence the terminology *near-horizon limit*). Once again, in that limit the constant part of H_p becomes irrelevant.

3. Lastly, and most importantly, this limit has an interesting interpretation from the microscopic D-brane point of view as a field theory limit in which the world-volume theory on the brane decouples from the bulk gravity.

   In particular, for the D3-brane one takes $\alpha' \rightarrow 0$ while keeping the energy E of stretched strings between the branes, $E \sim \alpha'/r$, fixed. One sees that this amounts to taking $r \rightarrow 0$, or, in light of the above, to considering a large number $N \sim Q_3$ of D3-branes, corresponding to a large-N limit of the $U(N)$ $\mathcal{N} = 4$ worldvolume super-Yang-Mills theory.

The near-horizon limits of the above solutions are

$$\begin{aligned} ds^2_{D3} &\rightarrow \frac{r^2}{Q_3^{1/2}}d\vec{x}^2 + \frac{Q_3^{1/2}}{r^2}dr^2 + Q_3^{1/2}d\Omega_5^2 && (6.23)\\ ds^2_{M2} &\rightarrow \frac{r^4}{Q_2^{2/3}}d\vec{x}^2 + \frac{Q_2^{1/3}}{r^2}dr^2 + Q_2^{1/3}d\Omega_7^2 && (6.24)\\ ds^2_{M5} &\rightarrow \frac{r}{Q_5^{1/3}}d\vec{x}^2 + \frac{Q_5^{2/3}}{r^2}dr^2 + Q_5^{2/3}d\Omega_4^2 && (6.25)\end{aligned}$$

We see that in each case we obtain a factorized (i.e. product) geometry, the r-dependence having disappeared from the metric of the transverse sphere.

Comparing with (6.13), we see that from the D3-brane solution we have obtained in the near-horizon limit the geometry

$$\mathrm{D3} \to AdS_5 \times S^5 \tag{6.26}$$

with the curvature radii R_{AdS} and R_S of the two factors given by

$$R_{AdS} = R_S = Q_3^{1/4} \sim \ell_s (g_s N)^{1/4} \ . \tag{6.27}$$

This metric (supplemented by the appropriate limit of the RR five-form field strength) is a famous and important solution of tpye IIB supergravity. In particular, this solution is *maximally supersymmetric*, i.e. it has as many unbroken supersymmetries as Minkowski space, namely 32 supercharges (in contrast with the D3-metric which preserves only half the supersymmetries). In particular, we thus find an enhancement of supersymmetry in the near-horizon limit.

The above considerations thus suggest (admittedly very tentatively at this point) a possible relation between string theory on $AdS_5 \times S^5$ and large-N super-Yang-Mills theory. This relation will be the subject of the lectures by Massimo Bianchi on the AdS/CFT correspondence.

From the M2- and M5-branes we also obtain geometries of the form $AdS \times S$ in the limit $r \to 0$. To see this, for the M2-brane we change coordinates,

$$r^2 = 2 Q_2^{1/6} \rho \ , \tag{6.28}$$

to find

$$\mathrm{M2} \to AdS_4 \times S^7 \tag{6.29}$$

with

$$2 R_{AdS} = R_S = Q_2^{1/6} \ . \tag{6.30}$$

Likewise, for the M5-brane, with the change of coordinates

$$r = \rho^2 / (4 Q_5^{1/3}) \ , \tag{6.31}$$

one finds

$$\mathrm{M5} \to AdS_7 \times S^4 \tag{6.32}$$

with

$$R_{AdS} = 2 R_S = 2 Q_5^{1/3} \ . \tag{6.33}$$

These are maximally supersymmetric solutions of eleven-dimensional supergravity (and the former has been studied intensively and extensively in the context of Kaluza-Klein supergravity in the 1980's).

We see that one way of interpreting these non-dilatonic brane solutions is as *interpolating solitons*, namely as half-supersymmetric soliton configurations which interpolate between the maximally supersymmetric Minkowski vacuum at $r \to \infty$ and the maximally supersymmetric $AdS \times S$ vacua at $r \to 0$. This is akin to the behaviour of solitons in other field theories, e.g. the kink solution of sine-Gordon theory.

6.3 Problems

1. Show that the AdS-metric in global coordinates (6.10) can also be written in the equivalent form

$$ds^2 = R_{AdS}^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2) \tag{6.34}$$

2. Using (6.28) and (6.31), verify (6.29,6.30) and (6.32,6.33) respectively.

3. Show that the near-horizon limit of the D1$\|$D5 metric (5.7) is $AdS_3 \times S^3 \times \mathbb{R}^4$. What are the curvature radii in terms of Q_1 and Q_5?

7 References: Review Articles

7.1 Generalizations and Omissions

In these lectures we have just considered the most elementary p-brane solutions of supergravity known, depending on a single antisymmetric tensor field and a single scalar. There are many other solutions in $d = 10$ and $d < 10$ describing more complex situations: branes at angles, configurations describing various bound states of branes, multi-string and -brane junctions, networks constructed from them, supersymmetric configurations of branes wrapped over 'cycles' more complicated than products of circles, non-supersymmetric configurations, etc.

For more information on the use of solitonic branes for the study of black holes in string theory, and also for the important application to the AdS/CFT correspondence, you are referred to the list of references below.

7.2 General Reviews of Non-Perturbative String Theory and String Dualities

1. J.H. Schwarz, *Lectures on Superstring and M Theory Dualities*, `hep-th/9607201`
2. J. Polchinski, *TASI Lectures on D-Branes*, `hep-th/9611050`
3. P.K. Townsend, *Four Lectures on M-Theory*, `hep-th/9612121`
4. A. Sen, *An Introduction to Non-perturbative String Theory*, `hep-th/9802051`
5. O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, *Large N Field Theories, String Theory and Gravity*, `hep-th/9905111`
6. E. Kiritsis, *Supersymmetry and Duality in Field Theory and String Theory*, `hep-ph/9911525`
7. D. Marolf, *String/M-branes for Relativists*, `gr-qc/9908045`
8. M. Duff, *TASI Lectures on Branes, Black Holes and Anti-de Sitter Space*, `hep-th/9912164`
9. C.V. Johnson, *D-Brane Primer*, `hep-th/0007170`
10. A.W. Peet, *TASI lectures on black holes in string theory*, `hep-th/0008241`

7.3 Review Articles on Supergravity and String Theory Solitons

1. C.G. Callan, Jr., J.A. Harvey, A. Strominger, *Supersymmetric String Solitons*, `hep-th/9112030`
2. M.J. Duff, R.R. Khuri, J.X. Lu, *String Solitons*, `hep-th/9412184`
3. J.M. Maldacena, *Black Holes in String Theory*, `hep-th/9607235`
4. E. Bergshoeff, *p-Branes, D-branes amd M-branes*, `hep-th/9611099`
5. M.J. Duff, *Supermembranes*, `hep-th/9611203`
6. K.S. Stelle, *Lectures on Supergravity p-Branes*, `hep-th/9701088`
7. D. Youm, *Black Holes and Solitons in String Theory*, `hep-th/9710046`
8. K.S. Stelle, *BPS Branes in Supergravity*, `hep-th/9803116`
9. R. D'Auria, P. Fre, *BPS Black Holes in Supergravity*, `hep-th/9812160`
10. K. Skenderis, *Black holes and branes in string theory*, `hep-th/9901050`